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THE EQUATIONS OF SOFT-FERROMAGNETIC ELASTIC PLATES

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Abstract—The aim of this paper is a sound formulation of coupled magnetomechanical equations which describe elastic plates made of a soft-ferromagnetic material. This is achieved by starting from a general formulation by means of the principle of virtual power within a nonlinear framework and specializing the approach by an adequate selection of the virtual fields in interest. All mechanical boundary conditions, involved as they may be, are obtained simultaneously with the plate equations. Constitutive equations are obtained on a thermodynamical basis. Finally, the equations which govern magnetomechanical perturbations about a rigid-body solution with a normal bias magnetic field are deduced with a view to studying magnetoelastic stability problems. Not only does the bias field alter the value of the flexural rigidity of the plate, but it also breaks the material symmetry to yield a complex magnetomechanical coupling. This materializes in the fact that the two-dimensional mechanical plate problem couples in a nontrivial manner with the magnetic problem which remains tri-dimensional but whose solution intervenes in the plate equation via averages throughout the thickness of the plate.

1. INTRODUCTION

The buckling of elastic structures is a most cultivated subject matter. Within the last fifteen years, however, a dimension has been added to such a problem by considering the effect of a magnetic field whenever the structures possess the internal mechanism to respond to such a physical loading. This is the case of ferromagnetic materials used in the design of powerful electromagnets such as those used for the acceleration of plasmas in controlled-fusion power development. It is expected that the mechanical structure considered, for instance schematized by an elastic plate or an elastic beam, will deflect in an unstable manner once a sufficiently intense applied, or self produced, magnetic field arises. The critical magnitude of the field reached in these conditions is the buckling value of a *magnetoelastic buckling problem*. This may explain the mechanical breakdown of certain structures, not under pure mechanical loadings, but under the action of a very intense magnetic field, the intermediate agent between this field and the mechanical behavior being a magnetic force. Recent works in this field of study using various schematizations are reviewed by Moon[1]. Works of special interest in this context are the pioneer ones of Moon and Pao[2, 3], who consider the action of ponderomotive forces and couples on an otherwise classical linear elastic body, neglect edge effects and show in these conditions that the dynamical and statical buckling values are equal, the work of Wallerstein and Peach[4], who account for the finiteness of the plate for the determination of the magnetic field inside the plate, a work by Pao and Yeh[5], who use somewhat general magnetoelastic equations but who do not formulate boundary conditions, another work by Dalrymple *et al.*[6]—who determine the buckling value of a half-restrained elliptical plate deflected in a cylindrical mode by use of the Moon–Pao approach and, more recently, more comprehensive treatments by Van de Ven[7–8] and Parkus[9], which treatments suffer, nonetheless, from certain limitations.

It is the need to determine the critical buckling value for magnetoelastic plates of *various shapes* submitted to *various* types *of mechanical boundary conditions* which provides the impetus for the present study. This aim, however, will be attained only in a companion paper for we desire first to settle in one place and in a somewhat rational manner the coupled mechanical-magnetic field equations, constitutive equations and boundary conditions which govern structures of the magnetoelastic-plate type. Although we shall naturally specialize to the case of *thin plane plates*, we intend to approach the looked for solution in the most general framework. Indeed, while a thin plate is usually considered as a two-dimensional structure (the thickness $2h$ of the plate is much smaller than the width and length of the plate), the problem related to magnetic equilibrium of the plate remains a three-dimensional one. The basic idea therefore is to start from a fully three-dimensional theory for both mechanical and magnetic descriptions and then to go over an essentially two-dimensional description for mechanical effects by means of an average throughout the thickness of the plate. Furthermore, in order to study buckling effects one must study the stability by small variations superimposed on *finite* fields (here a finite initial magnetic field). As already shown in various wave-type problems (e.g. [10]), this can be achieved in all rigor only if one starts from a fully nonlinear theory. Otherwise, one may incidentally discard several magnetoelastic couplings. In the present approach this is achieved by starting from a formulation by means of the *principle of virtual power* within a *nonlinear* framework. This type of formulation presents several advantages. First, we already have at hand a good formulation of the theory of *hard-ferromagnetic* (hence necessarily *nonlinear*) deformable bodies by means of the principle of virtual power[11, 12]. This can be specialized, still in a three-dimensional framework, to the case of *soft-ferromagnetic* bodies. Soft-ferromagnetic bodies are those magnetic bodies which behave linearly for weak applied magnetic fields, naturally start to behave nonlinearly for relatively strong magnetic fields, and finally saturate but, in contrast with hard-ferromagnetic bodies, they do not present hysteresis effects and ferromagnetic exchange effects (which are responsible for ferromagnetic ordering) can be neglected in them. Then the advantages of the virtual-power formulation are that (i) *boundary conditions*, involved as they may be, follow on an equal footing with the corresponding field equations (this is important in plate theory where the question of boundary conditions is always tediously dealt with) and (ii) by selecting at will the structure of the virtual velocity field, one may place in evidence any of the commonly accepted plate theories (compare Germain[13]). In doing so in Section 2 we shall obtain the nonlinear equations which govern soft-ferromagnetic nonlinear elastic thin plates within the Kirchhoff–Love framework, whatever the contour of the plate, which may in fact present angulous points. All statically admissible boundary conditions and kinematically admissible ones thanks to the duality inherent in the virtual-power formulation, follow without ambiguity. Having considered a three-dimensional nonlinear isotropic elastic-magnetic behavior (Section 3), we recall in Section 4 in which conditions is a mechanical equilibrium reached in a uniformly magnetized body in a rigid-body state. Then, in Section 5, a linearization, by infinitesimal variation about this state of all field, constitutive and boundary equations, yields the *bending equations* for small deflections of the plate, which will be exploited in the companion paper. Remarkable features show up during the variational procedure. These features are common to all linearization procedures performed about a state in which there exists a finite bias electromagnetic field (compare Maugin[14] for hard ferromagnets and Maugin and Pouget[15] for elastic ferroelectrics—see also the review paper by Maugin[16]). Not only coupled mechanical-magnetic effects occur through the magnetic body force and couple, but the intense initial bias magnetic field has for effect first to alter to some extent (a few per cent) the value of elastic coefficients (so-called *stiffening*), and next to induce coupled magnetoelastic properties, such as *piezomagnetism*, which would be nonexistent if the linearization was performed about a natural field-free state. In other words, the initial magnetic field breaks the *ideal symmetry* (here isotropy) to produce a weaker material symmetry which allows for coupled magnetoelastic properties which, otherwise, are not admissible. This finally results in the fact that perturbations take place in a state of *induced anisotropy*. With an initially transverse (to the plate) magnetic field, the body thus acquires uniaxial (material) symmetry with respect to the normal to the plate plane. In the present case this has for effect to couple the essentially two-dimensional mechanical bending problem with the three-dimensional magnetic one. The involved system of governing equations thus obtained is commented upon and compared to previous proposals—which discarded the above-described effects, in the final Section 6. The buckling of circular and rectangular plates will be examined in the companion paper on the basis of this system.

2. FORMULATION VIA THE PRINCIPLE OF VIRTUAL POWER

In order to determine the bending equations within the largest possible framework we shall start from the static form of the principle of virtual power for a general three dimensional nonlinear soft-ferromagnetic body. In a general manner, in the absence of inertial forces, we have (see Maugin[12])

$$\mathcal{P}_i^*(\mathcal{D}) + \mathcal{P}_e^*(\bar{\mathcal{D}}) = 0, \tag{2.1}$$

where $\mathcal{P}_i^*$ represents the total virtual power of *internal forces* (e.g. stresses) and $\mathcal{P}_e^*$ represents the total virtual power of *external* forces, acting at a distance within the volume $\mathcal{D}$, or by contact at the regular boundary $\partial\mathcal{D}$. Here $\mathcal{D}$ denotes the *open* simply connected region of physical space E^3 occupied by the body at time t in its current configuration K_t. $\bar{\mathcal{D}}$ is the closure of the set $\mathcal{D}$. The boundary $\partial\mathcal{D}$ is equipped with unit outward normal $\mathbf{n}$. In a general manner, an asterisk denotes a virtual field or the value taken by an expression in such a field. For a *soft ferromagnetic* body in which both gyromagnetic effects and exchange-ferromagnetic effects (see Maugin[11] for these effects) can be discarded, the two quantities present in (2.1), in the absence of body force of nonelectromagnetic origin, are given by (see Maugin[11 or 12]; tr = trace)

$$\mathcal{P}_i^*(\mathcal{D}) = -\int_{\mathcal{D}} [\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D}^*) - \rho\,{}^L\mathbf{B}\cdot\mathbf{m}^*]\,\mathrm{d}v \tag{2.2}$$

and

$$\mathcal{P}_e^*(\bar{\mathcal{D}}) = \int_{\mathcal{D}} (\mathbf{f}^{em}\cdot\mathbf{v}^* + \rho\mathbf{B}\cdot\dot{\boldsymbol{\mu}}^*)\,\mathrm{d}v + \int_{\partial\mathcal{D}} (\mathbf{T}+\mathbf{T}^{em})\cdot\mathbf{v}^*\,\mathrm{d}a, \tag{2.3}$$

where ρ is the matter density in K_t, $\boldsymbol{\sigma} = \{\sigma_{ij} = \sigma_{ji}\}$ is the *symmetric* intrinsic stress tensor (which reduces to the Cauchy stress tensor in the absence of magnetomechanical interactions), $\mathbf{B}$ is the Maxwellian magnetic induction, $\boldsymbol{\mu}$ is the magnetization per unit mass in K_t, $\mathbf{T}$ is the mechanical surface traction, $\mathbf{f}^{em}$ is the volume magnetic force, $\mathbf{T}^{em}$ is the corresponding magnetic surface traction, ${}^L\mathbf{B}$ is the so-called *local magnetic induction* (which accounts for the interactions between the magnetization field and the crystal lattice of the material), $\mathbf{D}$ is the rate-of-strain tensor and $\mathbf{m}$ is defined by

$$\mathbf{m} \equiv D_J\boldsymbol{\mu} = \dot{\boldsymbol{\mu}} - \boldsymbol{\Omega}\cdot\boldsymbol{\mu}, \tag{2.4}$$

where a superimposed dot indicates the material time derivative and $\boldsymbol{\Omega} = -\boldsymbol{\Omega}^T$ (T = transpose) is the rate-of-rotation tensor. In Cartesian tensor components ($i, j = 1, 2, 3$), v_i being the components of the velocity field of the continuum,

$$\begin{aligned} D_{ij} &= \frac{1}{2}(v_{i,j} + v_{j,i}) = v_{(i,j)},\ \Omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}) = v_{[i,j]}, \\ m_i &= \dot{\mu}_i - \Omega_{ij}\mu_j = \frac{\partial\mu_i}{\partial t} + v_j\mu_{i,j} - \Omega_{ij}\mu_j. \end{aligned} \tag{2.5}$$

For the time being we shall, however, most often use the direct dyadic notation without ambiguity. In reason of the lack of symmetry of certain tensors, it is important to note that the divergence of second-order tensors is taken on the first index, e.g. $(\mathrm{div}\,\mathbf{A})_i = A_{ji,j}$. The expression (2.2) is written as a linear continuous form on a set of *objective* virtual velocity ($\mathbf{D}$ and $\mathbf{m}$ are objective, i.e. transform as tensors of the correct order under *time-dependent* orthogonal transformations of the present configuration, if $\boldsymbol{\mu}$ is objective—see Maugin[17] for the objectivity of this last field). In general $\mathbf{B}$ and $\mathbf{H} = \mathbf{B} - \rho\boldsymbol{\mu}$, the magnetic field, satisfy Maxwell's equations while both fields $\boldsymbol{\sigma}$ and ${}^L\mathbf{B}$ require constitutive equations. We have, on $\partial\mathcal{D}$,

$$T_i^{em} = -[\![n_j t_{ji}^{em}]\!], \tag{2.6}$$

where the symbolism $[\![\ldots]\!]$ denotes the jump across $\partial\mathcal{D}$ and t_{ji}^{em} is the so-called magnetic stress tensor (see [12]). In eqn (2.3), $\dot{\boldsymbol{\mu}}^*$ denotes a virtual time variation of $\boldsymbol{\mu}$, so that the quantities $\mathbf{D}^*$

and $\mathbf{m}^*$ must be understood as

$$D^*_{ij} = \frac{1}{2}[(v^*_i)_{,j} + (v^*_j)_{,i}], \quad m^*_i = \dot{\mu}^*_i - \Omega^*_{ij}\mu_j \tag{2.7}$$

with

$$\Omega^*_{ij} = \frac{1}{2}[(v^*_i)_{,j} - (v^*_j)_{,i}] \tag{2.8}$$

for arbitrary fields $\mathbf{v}^*$ and $\dot{\boldsymbol{\mu}}^*$.

In a thin-plate theory where the coordinate along the thickness plays a privileged role in reason of the smallness of the thickness as compared to other dimensions of the plate, it proves convenient to introduce a *projection technique* as follows. Let $x_i = \{x_1, x_2, x_3;\ i = 1, 2, 3\}$ be a Cartesian coordinate system in such a way that $x_\alpha = \{x_1, x_2;\ \alpha = 1, 2\}$ be Cartesian coordinates in the medium plane Σ of the plate and x_3 be the normal coordinate to this plane. Let $\mathbf{n}_3$ be the unit normal to $\Sigma = \{x_3 = 0\}$. The *projection operator* onto Σ is defined by

$$\mathbf{P}_\Sigma = \mathbf{I} - \mathbf{n}_3 \otimes \mathbf{n}_3 \tag{2.9}$$

with

$$\mathbf{P}_\Sigma \cdot \mathbf{n}_3 = 0, \operatorname{tr} \mathbf{P}_\Sigma = 2, \quad \mathbf{P}_\Sigma{}^2 = \mathbf{P}_\Sigma, \tag{2.10}$$

$\mathbf{I}$ being the unit dyadic. Any three-dimensional tensor can be decomposed in a canonical manner with the help of $\mathbf{P}_\Sigma$. For instance, on account of the symmetry of $\mathbf{D}$ and the skewsymmetry of $\boldsymbol{\Omega}$,

$$\mathbf{D} = \mathbf{D}_\Sigma + (\mathbf{D} \otimes \mathbf{n}_3 + \mathbf{n}_3 \otimes \mathbf{D}) + D_{33}\mathbf{n}_3 \otimes \mathbf{n}_3 \tag{2.11}$$

and

$$\boldsymbol{\Omega} = \boldsymbol{\Omega}_\Sigma - (\boldsymbol{\Omega}_3 \otimes \mathbf{n}_3 - \mathbf{n}_3 \otimes \boldsymbol{\Omega}_3), \tag{2.12}$$

where

$$\begin{gathered} \mathbf{D}_\Sigma = \mathbf{P}_\Sigma(\mathbf{D}), \quad D_{33} = \mathbf{n}_3 \cdot (\mathbf{D} \cdot \mathbf{n}_3), \quad \mathbf{D} = \mathbf{P}_\Sigma(\mathbf{D} \cdot \mathbf{n}_3), \\ \boldsymbol{\Omega}_\Sigma = \mathbf{P}_\Sigma(\boldsymbol{\Omega}), \quad \boldsymbol{\Omega}_3 = -\mathbf{P}_\Sigma(\boldsymbol{\Omega} \cdot \mathbf{n}_3). \end{gathered} \tag{2.13}$$

More generally, we note

$$\mathbf{A}_\Sigma = \mathbf{P}_\Sigma(\mathbf{A}) \tag{2.14}$$

the full projection of a tensor $\mathbf{A}$ onto Σ, so that $\mathbf{n}_3$ is a zero vector of $\mathbf{A}_\Sigma$ with respect to all indices of the latter. We can therefore write the decompositions

$$\mathbf{v}^* = \mathbf{v}^*_\Sigma + v^*_3\mathbf{n}_3, \qquad v^*_3 = \mathbf{n}_3 \cdot \mathbf{v}^* \tag{2.15}$$

and

$$\begin{aligned} {}^L\mathbf{B} &= {}^L\mathbf{B}_\Sigma + {}^L\mathbf{B}_N\mathbf{n}_3, & {}^L\mathbf{B}_3 &= \mathbf{n}_3 \cdot {}^L\mathbf{B}, \\ \mathbf{f}^{em} &= \mathbf{f}^{em}_\Sigma + f^{em}_3\mathbf{n}_3, & f^{em}_3 &= \mathbf{n}_3 \cdot \mathbf{f}^{em}, \\ \mathbf{T}^{em} &= \mathbf{T}^{em}_\Sigma + T^{em}_3\mathbf{n}_3, & T^{em}_3 &= \mathbf{n}_3 \cdot \mathbf{T}^{em}, \\ \mathbf{T} &= \mathbf{T}_\Sigma + T_3\mathbf{n}_3, & T_3 &= \mathbf{n}_3 \cdot \mathbf{T}, \end{aligned} \tag{2.16}$$

as well as

$$\nabla = \nabla_\Sigma + \mathbf{n}_3 D, \quad \nabla_\Sigma = \mathbf{P}(\nabla), \quad D = \mathbf{n}_3 \cdot \nabla = \frac{\partial}{\partial x_3}.$$

We shall not need the decomposition of σ, as this will soon become evident.

The *Kirchhoff–Love* theory of thin-plates in the present context arises from assumptions made as regards the elements of decomposition, $\mathbf{v}_\Sigma^*$ and v_3^*, of v^*. Following Germain[13], we consider the following fields:

$$\begin{aligned} \mathbf{v}_\Sigma^* &= \mathbf{u}^*(x_\alpha) - x_3 \nabla_\Sigma w^*(x_\alpha), \quad \alpha = 1, 2, \\ v_3^*(x_\alpha) &= w^*(x_\alpha), \end{aligned} \tag{2.18}$$

where

$$\mathbf{u}^* \equiv \mathbf{u}_\Sigma^* = \mathbf{P}_\Sigma(\mathbf{u}^*), \tag{2.19}$$

or else

$$\mathbf{v}^*(x_i) = \mathbf{u}_\Sigma^*(x_\alpha) - x_3 \nabla_\Sigma w^*(x_\alpha) + w^*(x_\alpha)\mathbf{n}_3. \tag{2.20}$$

It follows immediately fron eqn (2.20) that in the present plate theory

$$\mathbf{D}^* \equiv \mathbf{D}_\Sigma^*, \quad \mathbf{\Omega}_\Sigma^* = \frac{1}{2}[(\nabla_\Sigma \mathbf{u}_\Sigma^*)^T - (\nabla_\Sigma \mathbf{u}_\Sigma^*)], \quad \mathbf{\Omega}_3^* = \nabla_\Sigma w^*. \tag{2.21}$$

Therefore,

$$\operatorname{tr}(\sigma \mathbf{D}^*) = \operatorname{tr}(\sigma \mathbf{D}_\Sigma^*) = \operatorname{tr}(\sigma_\Sigma \mathbf{D}_\Sigma^*); \quad \sigma_\Sigma = \mathbf{P}_\Sigma(\sigma) \tag{2.22}$$

and

$$\mathbf{m}^* = \dot{\mu}^* - \mathbf{\Omega}^* \cdot \mu = \dot{\mu}^* - (\mathbf{\Omega}^* \cdot \mathbf{n}_3)\mu_3 - \mathbf{\Omega}^* \cdot \mu_\Sigma \tag{2.23}$$

with

$$\mu_3 = \mathbf{n}_3 \cdot \mu, \quad \mu_\Sigma = \mathbf{P}_\Sigma(\mu). \tag{2.24}$$

The last of eqns (2.22) says that we are considering *plane strains* in Σ. We can also remark that in making w^* intervene in both components of $\mathbf{v}^*$ simultaneously, we have imposed a *kinematical constraint* which can be stated in classical terms as: *the unit normal* $\mathbf{n}_3$ *remains normal to the deformed surface corresponding to the plane* $\{x_3 = 0\}$ *in the course of the deformation.* In these conditions

$$\int_{\mathscr{D}} \operatorname{tr}(\sigma \mathbf{D}^*)\, \mathrm{d}v = \int_{\partial\mathscr{D}} \mathbf{n}_\Sigma\, (\sigma_\Sigma \cdot \mathbf{v}_\Sigma^*)\, \mathrm{d}a - \int_{\mathscr{D}} (\operatorname{div}_\Sigma \sigma_\Sigma) \cdot \mathbf{v}_\Sigma^*\, \mathrm{d}v, \tag{2.25}$$

where $\mathbf{n}_\Sigma$ is the unit exterior normal to the body $\mathscr{D}$ on that part of $\partial\mathscr{D}$ which is orthognoal to Σ (hence the "contour" of the plate). Similarly, a brief calculation allows one to show that

$$\begin{aligned} \int_{\mathscr{D}} \rho^L \mathbf{B} \cdot (\mathbf{\Omega}^* \cdot \mu)\, \mathrm{d}v = &- \int_{\mathscr{D}} (\operatorname{div}_\Sigma \mathbf{t}_\Sigma^{\mathrm{int}}) \cdot \mathbf{v}_\Sigma^*\, \mathrm{d}v - \int_{\mathscr{D}} (\nabla_\Sigma \cdot \mathbf{p}_\Sigma) w^*\, \mathrm{d}v \\ &+ \int_{\partial\mathscr{D}} \mathbf{n}_\Sigma \cdot (\mathbf{t}_\Sigma^{\mathrm{int}} \cdot \mathbf{v}_\Sigma^*)\, \mathrm{d}a + \int_{\partial\mathscr{D}} (\mathbf{n}_\Sigma \cdot \mathbf{p}_\Sigma) w^*\, \mathrm{d}a, \end{aligned} \tag{2.26}$$

where we have set (A denotes the antisymmetry operation)

$$\mathbf{t}_\Sigma^{\text{int}} = \rho({}^L\mathbf{B}_\Sigma \otimes \boldsymbol{\mu}_\Sigma)^A = -(\mathbf{t}_\Sigma^{\text{int}})^T \tag{2.27}$$

and

$$\mathbf{p}_\Sigma = \rho(\mu_3{}^L\mathbf{B}_\Sigma - {}^L B_3 \mathbf{n}_\Sigma \tag{2.28}$$

Let S be the section of $\mathscr{D}$ by Σ and $\mathbf{n}_S$ the trace of $\mathbf{n}_\Sigma$ on Σ. ∂S is the border line of S in Σ and $\boldsymbol{\tau}_S$ is the unit tangent to ∂S oriented in the positive sense about $\mathbf{n}_3$. The set of edges (or, better, apices) that ∂S may present is denoted by the common symbol I (see Fig. 1). On carrying the expressions (2.25) and (2.26) in eqn (2.2) and accounting for the decompositions (2.16) in eqn (2.3), we obtain the following form for the principle of virtual power:

$$\begin{aligned}\int_{\mathscr{D}} [(\operatorname{div}_\Sigma \mathbf{t}_\Sigma + \mathbf{f}_\Sigma^{em}) \cdot \mathbf{v}_\Sigma^* - (\nabla_\Sigma \cdot \mathbf{p}_\Sigma - f_3^{em}) w^* + \rho({}^L\mathbf{B} + \mathbf{B}) \cdot \dot{\boldsymbol{\mu}}^*] \, \mathrm{d}v \\ - \int_{\partial\mathscr{D}} \{[\mathbf{n}_\Sigma \cdot \mathbf{t}_\Sigma - (\mathbf{T}_\Sigma + \mathbf{T}_\Sigma^{em})] \cdot \mathbf{v}_\Sigma^* - [(\mathbf{n}_\Sigma \cdot \mathbf{p}_\Sigma) + T_3 + T_3^{em}] w^*\} \, \mathrm{d}a = 0,\end{aligned} \tag{2.29}$$

where we have defined the (two-dimensional) *Cauchy stress tensor* $\mathbf{t}_\Sigma$ by

$$\mathbf{t}_\Sigma = \boldsymbol{\sigma}_\Sigma + \mathbf{t}_\Sigma^{\text{int}} = \boldsymbol{\sigma}_\Sigma - \rho({}^L\mathbf{B}_\Sigma \otimes \boldsymbol{\mu}_\Sigma)^A. \tag{2.30}$$

In general this tensor is *not* symmetric, except whenever $\boldsymbol{\mu}_\Sigma$ is parallel to ${}^L\mathbf{B}_\Sigma$. We now account for the fact that the thin plate has a thickness $2h$ and that $\mathbf{v}_\Sigma^*$ assumes the form $(2.18)_1$. We define the following *averaged* fields and moments which are functions of the x_α's only:

$$\begin{aligned}&\mathsf{N} = \int_{-h}^{+h} \mathbf{t}_\Sigma \, \mathrm{d}x_3, \quad \mathsf{M} = \int_{-h}^{+h} x_3 \mathbf{t}_\Sigma \, \mathrm{d}x_3, \\ &\mathsf{P} = \int_{-h}^{+h} \mathbf{p}_\Sigma \, \mathrm{d}x_3, \\ &\mathbf{f}_\Sigma = \int_{-h}^{+h} \mathbf{f}_\Sigma^{em} \, \mathrm{d}x_3, \quad f_3 = \int_{-h}^{+h} f_3^{em} \, \mathrm{d}x_3, \quad \mathbf{g}_\Sigma = \int_{-h}^{+h} x_3 \mathbf{f}_\Sigma^{em} \, \mathrm{d}x_3 \\ &\mathbf{F}_\Sigma = \int_{-h}^{+h} (\mathbf{T}_\Sigma + \mathbf{T}_\Sigma^{em}) \, \mathrm{d}x_3, \quad F_3 = \int_{-h}^{+h} (T_3 + T_3^{em}) \, \mathrm{d}x_3, \quad \mathscr{F}_\Sigma = \int_{-h}^{+h} x_3 (\mathbf{T}_\Sigma + \mathbf{T}_\Sigma^{em}) \, \mathrm{d}x_3.\end{aligned} \tag{2.31}$$

Then we immediately have the following auxiliary results:

$$\begin{aligned}\int_{\mathscr{D}} (\operatorname{div}_\Sigma \mathbf{t}_\Sigma + \mathbf{f}_\Sigma^{em}) \cdot \mathbf{v}_\Sigma^* \, \mathrm{d}_v = \int_S (\operatorname{div}_\Sigma \mathsf{N} + \mathbf{f}_\Sigma) \cdot \mathbf{u}_\Sigma^* \, \mathrm{d}\Sigma \\ - \int_S (\operatorname{div}_\Sigma \mathsf{M} + \mathbf{g}_\Sigma) \cdot \nabla_\Sigma w^* \, \mathrm{d}\Sigma;\end{aligned} \tag{2.32}$$

$$\begin{aligned}\int_S (\operatorname{div}_\Sigma \mathsf{M} + \mathbf{g}_\Sigma) \cdot \nabla_\Sigma w^* \, \mathrm{d}\Sigma = \int_{\partial S} \mathbf{n}_S \cdot (\operatorname{div}_\Sigma \mathsf{M} + \mathbf{g}_\Sigma) w^* \, \mathrm{d}l \\ - \int_S [\nabla_\Sigma \cdot (\operatorname{div}_\Sigma \mathsf{M} + \mathbf{g}_\Sigma)] w^* \, \mathrm{d}\Sigma;\end{aligned} \tag{2.33}$$

$$\int_{\mathscr{D}} (\nabla_\Sigma \cdot \mathbf{p}_\Sigma - f_3^{em}) w^* \, \mathrm{d}v = \int_S (\nabla_\Sigma \cdot \mathsf{P} - f_3) w^* \, \mathrm{d}\Sigma; \tag{2.34}$$

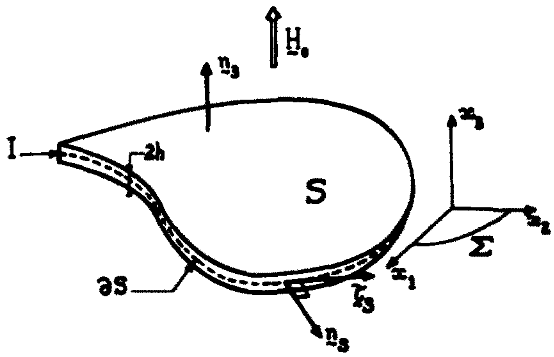

Fig. 1.

$$\int_{\partial\mathscr{D}}[(\mathbf{n}_\Sigma\cdot\mathbf{p}_\Sigma+T_3+T_3^{em}]w^*\,\mathrm{d}a=\int_{\partial S}(\mathbf{n}_S\cdot\mathsf{P}+F_3)w^*\,\mathrm{d}l \tag{2.35}$$

and

$$\int_{\partial\mathscr{D}}[\mathbf{n}_\Sigma\cdot\mathbf{t}_\Sigma-(\mathbf{T}_\Sigma+\mathbf{T}_\Sigma^{em})]\cdot\mathbf{v}_\Sigma^*\,\mathrm{d}a=\int_{\partial S}(\mathbf{n}_S\cdot\mathsf{N}-\mathbf{F}_\Sigma)\cdot\mathbf{u}_\Sigma^*\,\mathrm{d}l$$
$$-\int_{\partial S}(\mathbf{n}_s\cdot\mathsf{M}-\mathscr{F}_\Sigma)\cdot\nabla_\Sigma w^*\,\mathrm{d}l,$$

where $\mathrm{d}\Sigma$ is the unit surface element on Σ and $\mathrm{d}l$ is the unit arc along ∂S. For fields A defined at points on ∂S, we can write

$$\nabla_\Sigma A=\frac{\partial A}{\partial n_S}\mathbf{n}_S+\frac{\partial A}{\partial l}\boldsymbol{\tau}_S. \tag{2.37}$$

In particular, se set

$$\theta^*\equiv-\frac{\partial w^*}{\partial n_S}. \tag{2.38}$$

Then the last contribution in the r.h.s. of eqn (2.36) yields

$$\int_{\partial S}(\mathbf{n}_S\cdot\mathsf{M}-\mathscr{F}_\Sigma)\cdot\nabla_\Sigma w^*\,\mathrm{d}l=-\int_{\partial S}(\mathsf{M}_n-\mathscr{F}_\Sigma\cdot\mathbf{n}_S)\theta^*\,\mathrm{d}l \tag{2.39}$$
$$+\int_{\partial S}\frac{\partial}{\partial l}[(\mathsf{M}_\tau-\mathscr{F}_\Sigma\cdot\boldsymbol{\tau}_S)w^*]\,\mathrm{d}l$$
$$-\int_{\partial S}\frac{\partial}{\partial l}(\mathsf{M}_\tau-\mathscr{F}_\Sigma\cdot\boldsymbol{\tau}_S)w^*\,\mathrm{d}l,$$

where we have set

$$\mathsf{M}_n\equiv(\mathbf{n}_S\cdot\mathsf{M})\cdot\mathbf{n}_S,\quad \mathsf{M}_\tau=(\mathbf{n}_S\cdot\mathsf{M})\cdot\boldsymbol{\tau}_S. \tag{2.40}$$

The second line integral in eqn (2.39) yields zero except for the possible contributions at the set I of apices, which read $[\![\mathsf{M}_\tau-\mathscr{F}_\Sigma\cdot\boldsymbol{\tau}_S]\!]_I$ if $[\![\ldots]\!]$ here denotes the jump at an apex along ∂S.

Therefore, eqn (2.39) also reads

$$\int_{\partial S} (\mathbf{n}_S \cdot \mathsf{M} - \mathcal{F}_\Sigma) \cdot \nabla_\Sigma w^* \, dl = -\int_{\partial S} (\mathsf{M}_n - \mathcal{F}_\Sigma \cdot \mathbf{n}_S)\theta^* \, dl - \int_{\partial S} \frac{\partial}{\partial l}(\mathsf{M}_\tau - \mathcal{F}_\Sigma \cdot \tau_S) w^* \, dl + [\![\mathsf{M}_\tau - \mathcal{F}_\Sigma \cdot \tau_S]\!]_I w^*,$$

if the virtual field w^* is continuous at apices I along ∂S.

On collecting the auxiliary results (2.32)–(2.41) in eqn (2.29), we finally obtain the expression of the principle of virtual power in the present thin-plate theory:

$$\begin{aligned}
&\int_{\mathscr{D}} \rho({}^L\mathbf{B} + \mathbf{B}) \cdot \dot{\mu}^* \, dv + \int_S \{(\mathrm{div}_\Sigma \mathsf{N} + \mathbf{f}_\Sigma) \cdot \mathbf{u}_\Sigma^* + [\nabla_\Sigma \cdot (\mathrm{div}_\Sigma \mathsf{M} \\
&+ \mathbf{g}_\Sigma - \mathsf{P}) + f_3] w^*\} \, d\Sigma + \int_{\partial S} \{(\mathbf{F}_\Sigma - \mathbf{n}_S \cdot \mathsf{N}) \cdot \mathbf{u}_\Sigma^* + [F_3 - \mathbf{n}_S \cdot (\mathrm{div}_\Sigma \mathsf{M} \\
&+ \mathbf{g}_\Sigma - \mathsf{P}) - \frac{\partial}{\partial l}(\mathsf{M}_\tau - \mathcal{F}_\Sigma \cdot \tau_S)] w^* + (\mathcal{F}_\Sigma \cdot \mathbf{n}_S - \mathsf{M}_n)\theta^*\} \, dl \\
&+ [\![\mathsf{M}_\tau - \mathcal{F}_\Sigma \cdot \tau_S]\!]_I w^* = 0.
\end{aligned} \tag{2.42}$$

This is supposed to hold good for any element of volume, surface and line and for any field $\dot{\mu}^*$ in $\mathscr{D}$, any field $\mathbf{u}_\Sigma^*$ on Σ and along ∂S, any w^* on S, along ∂S and at apices I on ∂S, and any field θ^* along ∂S, from which there follows the following local equations:

$$^L\mathbf{B} + \mathbf{B} = \mathbf{0} \text{ in } \mathscr{D}, \tag{2.43}$$

$$\mathrm{div}_\Sigma \mathsf{N} + \mathbf{f}_\Sigma = \mathbf{0} \text{ on } S, \tag{2.44}$$

$$\nabla_\Sigma \cdot [\mathrm{div}_\Sigma \mathsf{M} + (\mathbf{g}_\Sigma - \mathsf{P})] + f_3 = 0 \text{ on } S, \tag{2.45}$$

$$\mathbf{F}_\Sigma - \mathbf{n}_S \cdot \mathsf{N} = \mathbf{0} \text{ along } \partial S - I, \tag{2.46}$$

$$\mathcal{F}_\Sigma \cdot \mathbf{n}_S - \mathsf{M}_n = 0 \text{ along } \partial S - I, \tag{2.47}$$

$$\mathbf{n}_S \cdot [\mathrm{div}_\Sigma \mathsf{M} + (\mathbf{g}_\Sigma - \mathsf{P})] + \frac{\partial}{\partial l}(\mathsf{M}_\tau - \mathcal{F}_\Sigma \cdot \tau_S) - F_3 = 0 \text{ along } \partial S - I, \tag{2.48}$$

and

$$[\![\mathsf{M}_\tau - \mathcal{F}_\Sigma \cdot \tau_S]\!] = 0 \text{ at } I \text{ along } \partial S. \tag{2.49}$$

The above-obtained equations can be interpreted in the following manner. The two-dimensional tensors N and M must be interpreted as the *tension–stress tensor* and the *bending–stress tensor*, respectively. The $\mathbf{f}_\Sigma$ and f_3 are a surface force acting in the plane of the plate and a surface density of normal forces, respectively. The scalar M_n must be interpreted as a *bending couple* because it is the thermodynamical dual quantity of θ^*, and the latter, according to eqn (2.38), is the virtual variation rate of the slope of the deformed plate at its contour. The quantities $\mathbf{F}_\Sigma$ and $(\mathbf{g}_\Sigma - \mathsf{P})$, in agreement with eqns (2.46) and (2.47), may be interpreted as a lineal force acting along ∂S and a lineal density of shearing force. In fact, eqn (2.48) includes the effects of a torsion couple—vie $\partial \mathsf{M}_\tau / \partial l$—and an effective shearing force. Finally, $[\![\mathcal{F}_\Sigma \cdot \tau_S]\!]$ is a finite force which is normal to the plane of the plate at angular points along ∂S. The tensor $\mathsf{K} = -\nabla_\Sigma(\nabla_\Sigma w^*)$

is the virtual rate of the curvature tensor of the deformed plate in a virtual motion. The problem encapsulated in the set (2.44)–(2.49) of mechanical field equations splits into two separate problems of which that one of interest in the present context is that of *pure bending* which is governed by the following equations:

$$\nabla_\Sigma \cdot [\mathrm{div}_\Sigma \mathsf{M} + (\mathbf{g}_\Sigma - \mathsf{P})] + f_3 = 0 \text{ on } S \tag{2.50}$$

with the accompanying boundary conditions along $\partial S - I$

$$\begin{aligned} &\mathbf{n}_s \cdot [\mathrm{div}_\Sigma \mathsf{M} + (\mathbf{g}_\Sigma - \mathsf{P})] + \frac{\partial}{\partial l}(\mathsf{M}_\tau - \mathcal{F}_\Sigma \cdot \tau_S) = F_3, \\ &\mathsf{M}_n = \mathcal{F}_\Sigma \cdot n_S \end{aligned} \tag{2.51}$$

and

$$[\mathsf{M}_\tau]_I = [\mathcal{F}_\Sigma \cdot \tau_S]_I \tag{2.52}$$

at angulous points along ∂S. In these equations the fields $\mathbf{f}_\Sigma, f_3$ and $\mathbf{g}_\Sigma$ are averages or moments of the magnetic body force $\mathbf{f}^{em}$ across the thickness of the plate. In general we have (for magnetostatics in insulators)

$$\mathbf{f}^{em} = \rho(\nabla \mathbf{B}) \cdot \boldsymbol{\mu}. \tag{2.53}$$

Similarly, the fields $\mathbf{F}_\Sigma$, F_3 and $\mathcal{F}_\Sigma$, whenever $\mathbf{T} = \mathbf{0}$ at $\partial\mathcal{D}$, are averages or moments of the magnetic surface force $\mathbf{T}^{em}$ across the thickness of the plate. Corresponding to eqn (2.53), we have (see Maugin [10])

$$\mathbf{T}^{em} = -\frac{1}{2}\mathbf{M}_\tau^2\mathbf{n}, \tag{2.54}$$

where $\mathbf{M}_\tau$ is the tangential component of the volume magnetization on the boundary of a magnetized three-dimensional body (zero magnetization outside the body). Finally, the field P (compare eqn 2.28) accounts for the effects of the local magnetic induction.

In addition to the above equations we must account for Maxwell's magnetostatic equations. In the configuration K_t and the plate being considered as a three-dimensional object $\mathcal{D}$ of boundary $\partial\mathcal{D}$, we have

$$\begin{aligned} &\nabla \times \mathbf{H} = 0 \text{ in } \mathcal{D} \\ &\nabla \cdot \mathbf{B} = 0 \text{ in } \mathcal{D}, \text{ with } \mathbf{H} = \mathbf{B} - \rho\boldsymbol{\mu} \end{aligned} \tag{2.55}$$

and

$$\nabla \times \mathbf{H} = \mathbf{0},\ \nabla \cdot \mathbf{H} = 0 \text{ outside } \mathcal{D}. \tag{2.56}$$

On the boundary,

$$\mathbf{n} \times [\mathbf{H}] = \mathbf{0},\ \mathbf{n} \cdot [\mathbf{B}] = 0. \tag{2.57}$$

Equations $(2.55)_1$ and (2.57) show that there exists a magnetic scalar potential Φ such that

$$\mathbf{H} = -\nabla\Phi, \tag{2.58}$$

so that eqns $(2.55)_2$ and $(2.57)_2$ take on the form

$$\nabla^2\Phi - \nabla \cdot (\rho\boldsymbol{\mu}) = 0 \text{ in } \mathcal{D},\ \nabla^2\Phi = 0 \text{ outside} \tag{2.59}$$

and

$$\left[\frac{\partial \Phi}{\partial n}\right] + M_n = 0 \text{ on } \partial\mathscr{D}, [\Phi] = 0 \tag{2.60}$$

where M_n is the normal component of the volume magnetization on $\partial\mathscr{D}$. In addition, Φ must go to zero at infinity from the body.

3. CONSTITUTIVE EQUATIONS

In order to close the above-obtained differential system we need constitutive equations for the fields $\boldsymbol{\sigma}$ and ${}^L\mathbf{B}$ or, equivalently, t and ${}^L\mathbf{B}$. We shall give in this section constitutive equations for a nonlinear elastic three-dimensional body. In the absence of dissipative processes and thermal effects the Clausius–Duhem inequality given in Maugin[11] reduces to the equality (here we have *real* fields, hence *no* asterisks):

$$\rho\dot{\psi} = \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D}) - \rho{}^L\mathbf{B}\cdot\mathbf{m}, \tag{3.1}$$

where ψ is the free energy per unit mass in the configuration K_t. On account of eqn (2.4) we can also write

$$\rho\dot{\psi} = t_{ji}v_{i,j} - \rho{}^LB_i\dot{\mu}_i \tag{3.2}$$

where (compare eqn 2.30)

$$t_{ji} = \sigma_{ji} - \rho{}^LB_{[j}\mu_{i]}. \tag{3.3}$$

Let a general nonlinear deformation between a natural reference free state or configuration K_R and K_t be described by the motion mapping

$$x_i = \mathscr{X}_i(X_K, t), \tag{3.4}$$

where $X_K, K = 1, 2, 3,$ are Cartesian material coordinates. With

$$x_{iK} = \frac{\partial \mathscr{X}_i}{\partial X_K}, \quad J = \det|x_{iK}| > 0, \quad X_{Kj} = \frac{\partial X_K}{\partial x_j}, \tag{3.5}$$

we can define the following *convected* fields (i.e. fields pulled back in the reference configuration):

$${}^Lt_{Ki} = J\,X_{Kj}t_{ji}, \quad {}^L\mathscr{B}_K = J\,X_{Ki}{}^LB_i, \quad \nu_K = \mu_jx_{jK}. \tag{3.6}$$

Reciprocally,

$$t_{ji} = J^{-1}x_{jK}{}^Lt_{Ki}, \quad {}^LB_i = J^{-1}x_{iK}{}^L\mathscr{B}_K, \quad \mu_j = \nu_KX_{Kj}. \tag{3.7}$$

The tensor ${}^Lt_{Ki}$ is none other than the first Piola–Kirchhoff stress tensor. With ρ_0 the matter density in K_R, we have

$$\rho_0 = \rho J. \tag{3.8}$$

Then, on account of eqns (3.7) and (3.8), eqn (3.2) yields

$$\rho_0\dot{\psi} = {}^Lt_{Ki}V_{i,K} - \rho{}^L\mathscr{B}_K\dot{\mu}_ix_{iK}, \tag{3.9}$$

if $V_i = v_i(X_K, t)$ is the velocity function expressed in terms of X_K and t. From the last of eqns

(3.6) we deduce that

$$\dot{\mu}_j x_{jK} = \dot{\nu}_K - \mu_i V_{i,K}, \tag{3.10}$$

so that eqn (3.9) transforms to

$$\rho_0 \dot{\psi} = \tilde{t}_{Ki} V_{i,K} - \rho^L \mathcal{B}_K \dot{\nu}_K, \tag{3.11}$$

where

$$\tilde{t}_{Ki} = {}^L t_{Ki} + \rho^L \mathcal{B}_K \mu_i \text{ or } {}^L t_{Ki} = \tilde{t}_{Ki} - \rho^L \mathcal{B}_K \mu_i. \tag{3.12}$$

For a *nonlinear elastic* body one takes

$$\psi = \bar{\psi}(x_{iK}, \nu_K), \tag{3.13}$$

so that eqn (3.11), posited to be valid for any $V_{i,K}$ and $\dot{\nu}_K$, yields the constitutive equations

$$\tilde{t}_{Ki} = \rho_0 \frac{\partial \bar{\psi}}{\partial x_{iK}}, \quad {}^L\mathcal{B}_K = -J \frac{\partial \bar{\psi}}{\partial \nu_K}. \tag{3.14}$$

An *objective* (i.e. rotationally invariant) form of (3.13) is

$$\psi = \hat{\psi}(E_{KL}, \nu_K); \quad E_{KL} \equiv \frac{1}{2}(x_{iK} x_{iL} - \delta_{KL}) = E_{LK}, \tag{3.15}$$

so that

$$\tilde{t}_{Ki} = \rho_0 \frac{\partial \hat{\psi}}{\partial E_{KL}} x_{iL}, \quad {}^L\mathcal{B}_K = -J \frac{\partial \hat{\psi}}{\partial \nu_K}. \tag{3.16}$$

Returning to ${}^L t_{Ki}$ via the second of eqns (3.12), we have the constitutive equations

$${}^L t_{Ki} = \rho_0 \left(\frac{\partial \hat{\psi}}{\partial E_{KL}} x_{iL} + \frac{\partial \hat{\psi}}{\partial \nu_K} \mu_i \right), \quad {}^L\mathcal{B}_K = -J \frac{\partial \hat{\psi}}{\partial \nu_K}, \tag{3.17}$$

and, through eqns (3.7),

$$t_{ji} = \rho \left(\frac{\partial \hat{\psi}}{\partial E_{KL}} x_{iL} + \frac{\partial \hat{\psi}}{\partial \nu_K} \mu_i \right) x_{jK}, \quad {}^L B_i = -\frac{\partial \hat{\psi}}{\partial \nu_K} x_{iK}. \tag{3.18}$$

For a body which behaves *isotropically* with respect to K_R, the functional dependence (3.15) reduces to (see the list of invariants in p. 1079 in Maugin[10])

$$\psi = \psi(I_\alpha; \quad \alpha = 1, 2, \ldots, 6) \tag{3.19}$$

with

$$\begin{gathered} I_1 = \operatorname{tr} \mathbf{E}, \quad I_2 = \operatorname{tr} \mathbf{E}^2, \quad I_3 = \operatorname{tr} \mathbf{E}^3, \\ I_4 = \boldsymbol{\nu}^2, \quad I_5 = \boldsymbol{\nu} \cdot (\mathbf{E} \cdot \boldsymbol{\nu}), \quad I_6 = \boldsymbol{\nu} \cdot (\mathbf{E}^2 \cdot \boldsymbol{\nu}). \end{gathered} \tag{3.20}$$

4. UNIFORMLY MAGNETIZED RIGID-BODY SOLUTION

A three-dimensional solution corresponding to a uniformly magnetized rigid-body state is noted

$$S_0 = \{\mathbf{x} = \mathcal{X}(\mathbf{X}) = \mathbf{X}, \quad \boldsymbol{\mu}_0 = \boldsymbol{\mu}, \quad \mathbf{H} = \mathbf{H}_0; \rho_0\}, \tag{4.1}$$

with

$$(x_{iK})_0 = \delta_{iK}, \quad (E_{KL})_0 = 0, \quad (\nu_K)_0 = \mu_{0i}\delta_{iK}, \quad \nabla \boldsymbol{\nu}_0 = \mathbf{0}, \quad \nabla \mathbf{H}_0 = \mathbf{0}.$$

Maxwell's equations are automatically satisfied by spatially uniform fields. We neglect demagnetizing effects so that the three-dimensional body in the *initial* configuration K_0 corresponding to the solution (4.1) may have any regular shape. On account of eqn (3.19) the constitutive equations (3.18) yield *initial* stresses $\mathbf{t}_0$ and an initial local magnetic induction ${}^L\mathbf{B}_0$ as (see Maugin[10], pp. 1082–1083)

$$t_{0ji} = t_1\delta_{ji} + (t_2 + \hat{\chi}_0 M_0^2)d_j d_i, \quad d_i = \mu_{0i}/|\boldsymbol{\mu}_0|, \quad {}^L B_{0i} = -\hat{\chi}_0 M_{0i} \tag{4.3}$$

with

$$t_1 = \rho_0\left(\frac{\partial\psi}{\partial I_1}\right)_0, \quad t_2 = \rho_0\mu_0^2\left(\frac{\partial\psi}{\partial I_5}\right)_0, \quad \hat{\chi}_0 = 2\rho_0^{-1}\left(\frac{\partial\psi}{\partial I_4}\right)_0. \tag{4.4}$$

Clearly, K_0 is not a natural free-field configuration and should therefore not be mistaken for K_R. This is essential in looking for perturbations about K_0. Although $t_{0ji,j} = 0$ in $\mathscr{D}$, it can be shown that mechanical equilibrium of the three-dimensional body is guaranteed only if surface tractions $\mathbf{t}_{(n)0}$ are applied on $\partial\mathscr{D}$ at K_0; $\mathbf{t}_{(n)0}$ satisfies the condition (see Maugin[10])

$$\left(t_1 + \frac{1}{2}M_0^2\right)\mathbf{n}_0 + (t_2 + \hat{\chi}_0 M_0^2)(\mathbf{n}_0 \cdot \mathbf{d})\mathbf{d} = \frac{1}{2}M_{0n}^2\mathbf{n}_0 + \mathbf{t}_{(n)0}, \tag{4.5}$$

which is a somewhat artificial situation. $\mathbf{n}_0$ is the unit outward normal to $\partial\mathscr{D}$ at K_0. As to the magnetic state of the body, the balance equation (2.43), on account of eqns (4.3), yields inside the body

$$\mathbf{H}_0 = (\hat{\chi}_0 - 1)\mathbf{M}_0, \tag{4.6}$$

so that the *initial magnetic susceptibility* of the body at K_0 is given by

$$\chi_0 = \frac{|\mathbf{M}_0|}{|\mathbf{H}_0|} = (\hat{\chi}_0 - 1)^{-1}. \tag{4.7}$$

In order to obtain the equations governing small perturbations about the rigid-body solution S_0, we shall perform a so-called *Lagrangian* variation (see Maugin[10], p. 1084).

5. PERTURBATION EQUATIONS ABOUT S_0

In looking for perturbations about S_0 we shall only consider the pure bending problem in presence of magnetostatic fields in a soft-ferromagnetic elastic plate. Perturbations are defined by means of a variation δ of all field quantities. In particular, for three-dimensional fields we set

$$u_i = \delta\mathscr{X}_i, \quad \bar{\mu}_i = \delta\mu_i, \tag{5.1}$$

where u_i are the components of the three-dimensional displacement between S_0 and the infinitesimally close varied configuration. On account of eqns (4.2) and $(3.6)_3$, we immediately have

$$\begin{aligned} &\delta x_{iK} = u_{i,K} = u_{i,j}\delta_{jK}, \quad \delta J = \nabla \cdot \mathbf{u}, \quad \delta\rho = -\rho_0 \nabla \cdot \mathbf{u}, \\ &\delta X_{Ki} = -u_{j,i}\delta_{jK}, \delta E_{PQ} = \frac{1}{2}\delta_{Pp}\delta_{Qq}(u_{p,q} + u_{q,p}) \end{aligned} \tag{5.2}$$

and

$$\delta\nu_P = (\bar{\mu}_j + \mu_{0k}u_{k,j})\delta_{jP}. \tag{5.3}$$

We shall need the variation of the constitutive equations (3.18). After a somewhat lengthy calculation and having set

$$
\begin{aligned}
t_{ji}^0 &= \rho_0\left[\left(\frac{\partial\hat{\psi}}{\partial E_{KL}}\right)_0 \delta_{iL} - \frac{\partial\hat{\psi}}{\partial \nu_K}\mu_{0i}\right]\delta_{jK},\\
C_{jikl}^0 &= \rho_0\left(\frac{\partial^2\hat{\psi}}{\partial E_{PQ}\partial E_{KL}}\right)_0 \delta_{jK}\delta_{iL}\delta_{kP}\delta_{lQ} = C_{(ji)(kl)}^0 = C_{klji}^0,\\
\epsilon_{jik}^0 &= \left(\frac{\partial^2\hat{\psi}}{\partial E_{PQ}\partial \nu_K}\right)_0 \delta_{jP}\delta_{iQ}\delta_{kK} = \epsilon_{ijk}^0,\\
\chi_{jk}^0 &= \rho_0^{-1}\left(\frac{\partial^2\hat{\psi}}{\partial \nu_P\partial \nu_K}\right)_0 \delta_{jK}\delta_{kP} = \chi_{kj}^0,\\
{}^L B_{0i} &= -\left(\frac{\partial\hat{\psi}}{\partial \nu_K}\right)_0 \delta_{iK},
\end{aligned}
\tag{5.4}
$$

from eqns (3.18), (5.2) and (5.3), we obtain

$$\delta t_{ji} = \bar{C}_{jipq} u_{p,q} + \rho_0 \bar{E}_{jip}\bar{\mu}_p \tag{5.5}$$

and

$$\delta^L B_i = [{}^L B_{0q}\delta_{ip} - (\epsilon_{pqi}^0 + \chi_{qi}^0 M_{0p})] u_{p,q} - \rho_0 \chi_{ij}^0 \bar{\mu}_j \tag{5.6}$$

where

$$
\begin{aligned}
\bar{C}_{jipq} = {}& C_{jipq}^0 - t_{ji}^0\delta_{pq} + t_{iq}^0\delta_{jp} + (t_{jq}^0 + {}^L B_{0j} M_{0q})\delta_{ip}\\
&+ (\epsilon_{jiq}^0 M_{0p} + \epsilon_{pqj}^0 M_{0i}) + \chi_{jq}^0 M_{0i} M_{0p}
\end{aligned}
\tag{5.7}
$$

and

$$\bar{E}_{jip} = \epsilon_{jip}^0 - {}^L B_{0j}\delta_{ip} + \chi_{ip}^0 M_{0i}. \tag{5.8}$$

Whereas the tensors C_{jikl}^0, ϵ_{jik}^0 and χ_{ip}^0 are the second-order elasticity-coefficient tensor, the piezomagnetism-coefficient tensor and a magnetic-susceptibility tensor, evaluated at S_0, respectively, the tensors $\bar{C}_{jipq}$ and $\bar{E}_{jip}$ are *effective* material-coefficient tensors which account for the inital stresses and the presence of an initial magnetization at S_0. In the *isotropic case* corresponding to the functional dependence (3.19), we have (see Maugin[10]):

$$
\begin{aligned}
C_{jipq}^0 = {}& \lambda\,\delta_{ji}\delta_{pq} + \mu(\delta_{jp}\delta_{iq} + \delta_{jq}\delta_{ip}) + \alpha_1(\delta_{ji}d_q d_p + \delta_{pq}d_j d_i)\\
&+ \alpha_2(\delta_{jp}d_i d_q + \delta_{jq}d_i d_p + \delta_{ip}d_j d_q + \delta_{iq}d_j d_p) + \alpha_3 d_j d_i d_p d_q,\\
\epsilon_{jik}^0 = {}& M_0[g\delta_{ji}d_k + f(\delta_{jk}d_i + \delta_{ik}d_j) + d\mu_0^2 d_j d_i d_k],\\
\chi_{jk}^0 = {}& \hat{\chi}_0\delta_{jk} + \beta\, d_j d_k,
\end{aligned}
\tag{5.9}
$$

with

$$
\begin{aligned}
&\lambda = \rho_0\left(\frac{\partial^2\hat{\psi}}{\partial I_1^2}\right)_0,\quad \mu = \rho_0\left(\frac{\partial\hat{\psi}}{\partial I_2}\right)_0,\quad M_0 = \rho_0\mu_0,\\
&\alpha_1 = \rho_0\left(\frac{\partial^2\hat{\psi}}{\partial I_1 I_5}\right)_0,\quad \alpha_2 = \rho_0\left(\frac{\partial\hat{\psi}}{\partial I_6}\right)_0,\quad \alpha_3 = \rho_0\left(\frac{\partial^2\hat{\psi}}{\partial I_5^2}\right)_0\mu_0^2,
\end{aligned}
\tag{5.10}
$$

$$g = 2\left(\frac{\partial^2\hat{\psi}}{\partial I_1 \partial I_4}\right)_0, \quad f = \left(\frac{\partial\hat{\psi}}{\partial I_5}\right)_0, \quad d = 2\left(\frac{\partial^2\hat{\psi}}{\partial I_5 \partial I_4}\right)_0,$$

$$\hat{\chi}_0 = 2\rho_0^{-1}\left(\frac{\partial\hat{\psi}}{\partial I_4}\right)_0, \quad \beta = 2\rho_0^{-1}\left(\frac{\partial^2\hat{\psi}}{\partial I_4^2}\right)_0, \quad d_i = \mu_{0i}/|\mu_0|,$$

while t^0_{ji} and ${}^LB_{0i}$ are given by eqns (4.3) with the definitions (4.5). The material-coefficient tensors defined in eqns (5.9), as well as t^0_{ji} and ${}^LB_{0i}$, admit uniaxial symmetry with respect to the direction of the initial magnetization. In other words, the initial field $\mathbf{H}_0$ (with which $\mathbf{M}_0$ is aligned) has *broken the ideal symmetry* (here isotropy) so that perturbations occur about a state in which the material has acquired a weaker symmetry. This results in the fact that, insofar as perturbations are concerned, we have five elastic coefficients (isotropic ones, λ and μ, and three additional ones, α_1, α_2, α_3), two magnetic-susceptibility coefficients (the isotropic one $\hat{\chi}_0$ and the additional one β) and we have *induced piezomagnetism* (three coefficients, g, f and d) while piezomagnetism would not exist for a linear theory of isotropic material about a free state. On account of eqns (5.9), we obtain

$$\begin{aligned}\delta t_{ji} &= \bar{\lambda}\,\delta_{ji}(\nabla\cdot\mathbf{u}) + \bar{\mu}(u_{i,j}+u_{j,i}) + \bar{\alpha}_1\delta_{ji}Du_d + (\bar{\alpha}_1 - \bar{t}_2)(\nabla\cdot\mathbf{u})d_jd_i \\ &\quad + (\bar{\alpha}_2 + \bar{t}_2)d_iDu_j + \bar{\alpha}_3 d_iu_{d,j} + \bar{\alpha}_4 d_jDu_i + \bar{\alpha}_2 d_ju_{d,i} \\ &\quad + \bar{\alpha}_5 d_jd_iDu_d + \rho_oM_0[g\delta_{ji} + (\beta + d\mu_0^2)d_jd_i]\bar{\mu}_d \\ &\quad + \rho_0M_0(f+\hat{\chi}_0)d_i\bar{\mu}_j + \rho_0M_0(f-\hat{\chi}_0)\bar{\mu}_id_j,\end{aligned} \tag{5.11}$$

where we have set

$$\begin{aligned}&\bar{\lambda} = \lambda - t_1, \quad \bar{\mu} = \mu + t_1, \quad \bar{\alpha}_1 = \alpha_1 + gM_0^2, \quad \bar{t}_2 = t_2 + \hat{\chi}_0M_0^2, \\ &\bar{\alpha}_2 = \alpha_2 + fM_0^2, \quad \bar{\alpha}_3 = \alpha_2 + (\hat{\chi}_0 + 2f)M_0^2, \quad \bar{\alpha}_4 = \alpha_2 + t_2, \\ &\bar{\alpha}_5 = \alpha_3 + (\beta + 2d\mu_0^2)M_0^2\end{aligned} \tag{5.12}$$

and

$$u_d = \mathbf{u}\cdot\mathbf{d}, \quad \bar{\mu}_d = \mathbf{d}\cdot\bar{\mu}, \quad D = \mathbf{d}\cdot\nabla. \tag{5.13}$$

By the same token we have

$$\begin{aligned}\delta^LB_i &= -\rho_0\hat{\chi}_0\bar{\mu}_i - M_0[(\beta + d\mu_0^2)Du_d + g(\nabla\cdot\mathbf{u}) + \rho_0(\beta/M_0)\bar{\mu}_d]d_i \\ &\quad - M_0(f+\hat{\chi}_0)(Du_i + u_{d,i}).\end{aligned} \tag{5.14}$$

From here on we shall consider that, in K_0, $\mathbf{M}_0$ and $\mathbf{H}_0$ are directed along the normal to the plate, hence

$$\mathbf{d} = \mathbf{n}_{03}. \tag{5.15}$$

In varying eqn (2.50), we obtain

$$\nabla_\Sigma\cdot[\mathrm{div}_\Sigma(\delta\mathsf{M}) + (\delta\mathbf{g}_\Sigma - \delta\mathsf{P})] + \delta f_3 = 0. \tag{5.16}$$

We have

$$\begin{aligned}&\delta\mathsf{M}_{ij} = \int_{-h}^{+h} x_3\delta(P_{ik}P_{jl}t_{kl})\,\mathrm{d}x_3, \quad \delta g_i = \int_{-h}^{+h} x_3\delta(P_{ik}f^{em}_k)\,\mathrm{d}x_3, \\ &\delta\mathsf{P}_i = \int_{-h}^{+h}\delta[P_{ij}(\mathbf{p}_\Sigma)_j]\,\mathrm{d}x_3, \qquad \delta f_3 = \int_{-h}^{+h}\delta(\mathbf{n}_3\cdot\mathbf{f}^{em})\,\mathrm{d}x_3,\end{aligned} \tag{5.17}$$

if P_{ij} are the three-dimensional components of the essentially two-dimensional tensor $\mathbf{P}_\Sigma$. Theoretically we should account for the variation of $\mathbf{n}_3$ which intervenes in the various integrands in eqns (5.17) via δP_{ik} and $\delta \mathbf{n}_3$. But since the resulting variation must also be in Σ, and $\mathbf{f}_0^{em} = \mathbf{0}$, it is not difficult to see that

$$\begin{gathered} \mathbf{P}_\Sigma[\delta(P_{ik}P_{jl}t_{kl})] = P_{ik}P_{jl}\delta t_{kl}, \\ \mathbf{P}_\Sigma[\delta(P_{ik}f_k^{em})] = P_{ik}\delta f_k^{em}, \quad \mathbf{P}_\Sigma\delta[P_{ij}(\mathbf{p}_\Sigma)_j] = P_{ij}\delta(\mathbf{p}_\Sigma)_j, \\ \delta(\mathbf{n}_3\cdot\mathbf{f}^{em}) = \mathbf{n}_3\cdot\delta\mathbf{f}^{em} = \delta f_3^{em}. \end{gathered} \tag{5.18}$$

The projection onto Σ of eqns (5.11) and (5.14) yields

$$P_{kj}P_{li}\delta t_{ji} = P_{kj}P_{li}[\bar{\lambda}\,\delta_{ji}(\nabla\cdot\mathbf{u}) + \bar{\mu}(u_{i,j}+u_{j,i}) + \bar{\alpha}_1\delta_{ji}Du_3 + \rho_0 M_0 g\delta_{ji}\bar{\mu}_3] \tag{5.19}$$

and

$$P_{ki}\delta^L B_i = -P_{ki}[\rho_0\hat{\chi}_0\bar{\mu}_i + M_0(f+\hat{\chi}_0)(Du_i + u_{3,i})]. \tag{5.20}$$

This can also be written in the plate plane co-ordinates x_α, $\alpha = 1, 2$, as

$$\delta t_{\beta\alpha} = \bar{\lambda}\,\delta_{\beta\alpha}(\nabla\cdot\mathbf{u}) + \bar{\mu}(u_{\alpha,\beta}+u_{\beta,\alpha}) + \bar{\alpha}_1\delta_{\beta\alpha}Du_3 + \rho_0 M_0 g\,\delta_{\beta\alpha}\bar{\mu}_3 \tag{5.21}$$

and

$$\delta^L B_\alpha = -[\rho_0\hat{\chi}_0\bar{\mu}_\alpha + M_0(f+\hat{\chi}_0)(Du_\alpha + u_{3,\alpha})]. \tag{5.22}$$

From eqn (5.14) we also have

$$\delta^L B_3 = -\rho_0(\hat{\chi}_0+\beta)\bar{\mu}_3 - M_0[(\beta + d\mu_0{}^2) + 2(f+\hat{\chi}_0)]Du_3 - M_0 g(\nabla\cdot\mathbf{u}). \tag{5.23}$$

We have

$$\delta\mathbf{f}^{em} = \rho_0(\nabla(\delta\mathbf{B}))\cdot\boldsymbol{\mu}_0 = \rho_0(\nabla\mathbf{b})\cdot\boldsymbol{\mu}_0, \quad \mathbf{b} = \delta\mathbf{B}, \tag{5.24}$$

so that

$$\delta f_3^{em} = M_0 Db_3, \quad \delta f_\alpha^{em} = M_0 b_{3,\alpha}. \tag{5.25}$$

Finally,

$$\begin{aligned} \delta\mathbf{p}_\Sigma = {} & (\delta\rho)[\mu_{03}({}^L\mathbf{B}_\Sigma)_0 - {}^L B_{03}(\boldsymbol{\mu}_\Sigma)_0] \\ & + \rho_0[\bar{\mu}_3({}^L\mathbf{B}_\Sigma)_0 + \mu_{03}(\delta^L\mathbf{B}_\Sigma) - \delta^L B_3(\boldsymbol{\mu}_\Sigma)_0 - {}^L B_{03}\bar{\boldsymbol{\mu}}_\Sigma]. \end{aligned} \tag{5.26}$$

But, initially,

$$({}^L\mathbf{B}_\Sigma)_0 = 0, \quad (\boldsymbol{\mu}_\Sigma)_0 = 0, \quad \mu_{03} = \mu_0, \quad {}^L B_3 = -\hat{\chi}_0 M_0, \tag{5.27}$$

so that, in components in the plane Σ,

$$\begin{aligned} \delta p_\alpha &= M_0\delta^L b_\alpha + \rho_0\hat{\chi}_0 M_0\bar{\mu}_\alpha \\ &= -M_0^2(f+\hat{\chi}_0)(Du_\alpha + u_{3,\alpha}) \end{aligned} \tag{5.28}$$

after eqn (5.22). The displacement field must have the same structure as the virtual velocity field

(2.18), so that we take

$$u_i\begin{cases} u_\alpha = \theta_\alpha(x_\beta) - x_3\omega_{,\alpha}, & \alpha, \beta = 1, 2, \\ u_3 = \omega(x_\beta). \end{cases} \tag{5.29}$$

In these conditions,

$$\begin{gathered} Du_3 = 0, \quad \nabla \cdot \mathbf{u} = u_{\alpha,\alpha} = \theta_{\gamma,\gamma} - x_3\omega_{,\gamma\gamma}, \\ u_{\alpha,\beta} = \theta_{\alpha,\beta} - x_3\omega_{,\alpha\beta}, \quad u_{3,\alpha} = \omega_{,\alpha}, \quad Du_\alpha = -\omega_{,\alpha} \end{gathered} \tag{5.30}$$

since $D = \partial/\partial x_3$. Then eqns (5.21)–(5.23) and (5.28) take on the following form:

$$\delta t_{\beta\alpha} = \bar{\lambda}\,\delta_{\beta\alpha}(\theta_{\gamma,\gamma} - x_3\omega_{,\gamma\gamma}) + \bar{\mu}(\theta_{\alpha,\beta} + \theta_{\beta,\alpha} - 2x_3\omega_{,\alpha\beta}) + \rho_0 M_0 g \delta_{\beta\alpha}\bar{\mu}_3 \tag{5.31}$$

$$\delta^L B_\alpha = -\rho_0\hat{\chi}_0\bar{\mu}_\alpha, \tag{5.32}$$

$$\delta^L B_3 = -\rho_0(\hat{\chi}_0 + \beta)\bar{\mu}_3 - gM_0(\theta_{\gamma,\gamma} - x_3\omega_{,\gamma\gamma}), \tag{5.33}$$

and

$$\delta p_\alpha = 0. \tag{5.34}$$

This allows us to compute $\delta\,\mathsf{M}_{\beta\alpha}$, δg_α, $\delta\,\mathsf{P}_\alpha$ and δf_3 by integration throughout the thickness of the plate. We find thus

$$\delta\,\mathsf{M}_{\beta\alpha} = -\frac{2h^3}{3}(\bar{\lambda}\omega_{,\gamma\gamma}\delta_{\beta\alpha} + 2\bar{\mu}\omega_{,\alpha\beta}) + \rho_0 M_0 g\,\delta_{\beta\alpha}\int_{-h}^{+h} x_3\bar{\mu}_3\,\mathrm{d}x_3, \tag{5.35}$$

$$\delta g_\alpha = M_0\int_{-h}^{+h} x_3 b_{3,\alpha}\,\mathrm{d}x_3, \tag{5.36}$$

$$\delta\,P_\alpha = 0, \tag{5.37}$$

and

$$\delta f_3 = M_0\int_{-h}^{+h} Db_3\,\mathrm{d}x_3, \tag{5.38}$$

since $\bar{\mu}_3$ and b_3 are functions of all three co-ordinates (x_α, $\alpha = 1, 2$, and x_3). The bending equation (5.16) now yields

$$-\frac{2h^3}{3}(\bar{\lambda} + 2\bar{\mu})\Delta_\Sigma\Delta_\Sigma\omega + M_0[\Delta_\Sigma\hat{\hat{b}}_3 + \widehat{Db_3} + g\Delta_\Sigma\widehat{(\rho_0\bar{\mu}_3)}] = 0, \tag{5.39}$$

where Δ_Σ denotes the two-dimensional Laplacian operator in the plane Σ and we use the notation

$$\hat{A} = \int_{-h}^{+h} A\,\mathrm{d}x_3, \quad \hat{\hat{A}} = \int_{-h}^{+h} x_3 A\,\mathrm{d}x_3, \tag{5.40}$$

for fields $A = A(x_\alpha, x_3)$. The variation of Maxwell's magnetostatic equations in the volume of the plate yields

$$\nabla \times \mathbf{h} = \mathbf{0}, \quad \nabla \cdot \mathbf{b} = 0, \quad \mathbf{h} = \delta\mathbf{H}. \tag{5.41}$$

But, from $\mathbf{H} = \mathbf{B} - \rho\mathbf{u}$ and eqns (5.2), we have

$$\mathbf{h} = \mathbf{b} + \mathbf{M}_0(\nabla \cdot \mathbf{u}) - \rho_0\bar{\boldsymbol{\mu}}. \tag{5.42}$$

The first of eqns (5.41) shows that we can introduce a perturbation scalar magnetic potential φ such that $\mathbf{h} = -\nabla\varphi$. On account of this and eqns (5.30), from eqn (5.42) we deduce that

$$\rho_0\bar{\boldsymbol{\mu}} = \mathbf{b} + \mathbf{M}_0(\theta_{\gamma,\gamma} - x_3\omega_{,\gamma\gamma}) + \nabla\varphi. \tag{5.43}$$

The variation of eqn (2.43) yields

$$\mathbf{b} = -\,\delta^L\mathbf{B}. \tag{5.44}$$

On combining eqns (5.43), (5.44) and (5.32)–(5.33), we obtain

$$\begin{aligned} \rho_0\bar{\mu}_\alpha &= -\,\chi_0\varphi_{,\alpha}, \\ \rho_0\bar{\mu}_3 &= -\,\alpha[(1+g)M_0(\theta_{\gamma,\gamma} - x_3\omega_{,\gamma\gamma}) + D\varphi], \end{aligned} \tag{5.45}$$

where χ_0 has been defined in eqn (4.7) and we have set

$$\alpha = [(\hat{\chi}_0 + \beta) - 1]^{-1}. \tag{5.46}$$

It follows from the projection of eqn (5.43) along $\mathbf{n}_3$ that

$$b_3 = -\,M_0(\theta_{\gamma,\gamma} - x_3\omega_{,\gamma\gamma}) - D\varphi + \rho_0\bar{\mu}_3$$

or

$$b_3 = -\,(1+\alpha)D\varphi - M_0(1+\tilde{\alpha})(\theta_{\gamma,\gamma} - x_3\omega_{,\gamma\gamma}), \quad \tilde{\alpha} \equiv \alpha(1+g). \tag{5.47}$$

Equations $(5.45)_2$ and (5.47) allow us to evaluate the magnetic terms in eqn (5.39). We have

$$\hat{\hat{b}}_3 = \frac{2h^3}{3}M_0(1+\tilde{\alpha})\Delta_\Sigma\omega - (1+\alpha)\widehat{\widehat{D\varphi}}, \tag{5.48}$$

$$\widehat{Db_3} = \frac{2h^3}{3}\tilde{\alpha}M_0\Delta_\Sigma\Delta_\Sigma\omega - \alpha\Delta_\Sigma\widehat{D\varphi}. \tag{5.49}$$

and

$$\widehat{\widehat{(\rho_0\bar{\mu}_3)}} = \frac{2h^3}{3}\tilde{\alpha}M_0\omega_{,\gamma\gamma} - \widehat{\widehat{\alpha D\varphi}}. \tag{5.50}$$

Thus,

$$\Delta_\Sigma\widehat{\widehat{(\rho_0\bar{\mu}_3)}} = \frac{2h^3}{3}\tilde{\alpha}M_0\Lambda_\Sigma\Delta_\Sigma\omega - \alpha\Delta_\Sigma D\varphi. \tag{5.51}$$

Substituting now from eqns (5.48)–(5.51) into eqn (5.39), we arrive at the equation

$$\begin{aligned} &\mathscr{D}\Delta_\Sigma\Delta_\Sigma\omega + M_0[(1+\tilde{\alpha})(\Delta_\Sigma\widehat{\widehat{D\varphi}}) + (1+\alpha)(\widehat{D^2\varphi})] \\ &-2hM_0^2[(1+\tilde{\alpha})\Delta_\Sigma\omega] = 0, \end{aligned} \tag{5.52}$$

where we have defined an *effective flexural rigidity* of the plate by

$$\mathscr{D} = \frac{2h^3}{3}[(\bar{\lambda} + 2\bar{\mu}) - M_0^2(1 + \tilde{\alpha}(1+g))] = \frac{2h^3}{3}\left\{(\lambda + 2\mu) - M_0^2\left[(1 + \tilde{\alpha}(1+g)) - \frac{t_1}{M_0^2}\right]\right\}. \tag{5.53}$$

Maxwell's equation $(5.41)_2$ transforms in the same manner. On account of eqn (5.43), it renders

$$\nabla^2\varphi - \rho_0(\bar{\mu}_{\alpha,\alpha} + D\bar{\mu}_3) - M_0\Delta_\Sigma\omega = 0. \tag{5.54}$$

Accounting for eqns (5.45), we obtain thus

$$(1+\chi_0)\Delta_\Sigma\varphi + (1+\alpha)D^2\varphi = M_0(1+\bar{\alpha})\Delta_\Sigma\omega. \tag{5.55}$$

Insofar as mechanical boundary conditions are concerned, the *statically admissible* ones can be deduced by effecting the variation of eqns (2.51) and (2.52). That is, for the first two of these, if n_α and τ_α are the two-dimensional components of $\mathbf{n}_S$ and $\boldsymbol{\tau}_S$,

$$(\delta\mathbf{n}_S)\cdot[\mathrm{div}_\Sigma\,\mathsf{M} + (\mathbf{g}_\Sigma - \mathsf{P})]_0 + \mathbf{n}_{S0}\cdot[\mathrm{div}_\Sigma\delta\mathsf{M} + (\delta\mathbf{g}_\Sigma - \delta\mathsf{P})]$$

$$+(\delta\boldsymbol{\tau}_S)\cdot[\nabla_\Sigma(\mathsf{M}_\tau)]_0 + \frac{\partial}{\partial l_0}(\delta\mathsf{M}_\tau) = \delta F_3 + (\delta\boldsymbol{\tau}_S)\cdot[\nabla_\Sigma(\boldsymbol{\mathcal{F}}_\Sigma\cdot\boldsymbol{\tau}_S)]_0 + \frac{\partial}{\partial l_0}\delta(\boldsymbol{\mathcal{F}}_\Sigma\cdot\boldsymbol{\tau}_S) \tag{5.56}$$

and

$$\delta\,\mathsf{M}_n = \delta(\boldsymbol{\mathcal{F}}_\Sigma\cdot\mathbf{n}_S), \tag{5.57}$$

where

$$\frac{\partial}{\partial l} = \boldsymbol{\tau}_S\cdot\nabla_\Sigma, \quad \frac{\partial}{\partial l_0} = \boldsymbol{\tau}_{S0}\cdot\nabla_\Sigma, \tag{5.58}$$

and $\mathbf{n}_{S0}$ and $\boldsymbol{\tau}_{S0}$ are the unit normal and tangent to ∂S in its initial configuration ∂S_0. By the same token,

$$\delta\mathsf{M}_n = (\delta\mathbf{n}_S)\cdot(\mathsf{M}\cdot\mathbf{n}_S)_0 + \mathbf{n}_{S0}\cdot[(\delta\mathsf{M})\cdot\mathbf{n}_{S0}] + \mathbf{n}_{S0}\cdot[(\mathsf{M})_0\cdot\delta\mathbf{n}_S] \tag{5.59}$$

and

$$\delta\mathsf{M}_\tau = (\delta\mathbf{n}_S)\cdot(\mathsf{M}\cdot\boldsymbol{\tau}_S)_0 + \mathbf{n}_{S0}\cdot(\delta\mathsf{M}\cdot\boldsymbol{\tau}_{S0}) + \mathbf{n}_{S0}\cdot[(\mathsf{M})_0\cdot\delta\boldsymbol{\tau}_S]. \tag{5.60}$$

From eqns $(4.3)_1$, (2.40) and $(2.31)_1$, we find that

$$(\mathsf{M}_n)_0 = (\mathsf{M}_\tau)_0 = 0, \tag{5.61}$$

so that eqns (5.59)–(5.60) reduce to

$$\delta\mathsf{M}_n = \mathbf{n}_{S0}\cdot[(\delta\mathsf{M})\cdot\mathbf{n}_{S0}], \quad \delta\mathsf{M}_\tau = \mathbf{n}_{S0}\cdot[(\delta\mathsf{M})\cdot\boldsymbol{\tau}_{S0}]. \tag{5.62}$$

We also have

$$(\mathbf{g}_\Sigma)_0 = 0, \quad \mathsf{P}_0 = 0, \quad \delta F_3 = 0, \quad \delta\boldsymbol{\mathcal{F}}_\Sigma = \mathbf{0},$$

as is readily checked on account of eqn (2.54). The other initial contributions in eqn (5.56) vanish in reason of the spatial uniformity at S_0. It therefore remains

$$\mathbf{n}_{S0}\cdot[(\mathrm{div}_\Sigma\,\delta\mathsf{M}) + \delta\mathbf{g}_\Sigma] + \frac{\partial}{\partial l_0}(\delta\,\mathsf{M}_\tau) = 0, \quad \delta\,\mathsf{M}_n = 0 \tag{5.63}$$

on account of eqn (5.37). On substituting from eqns (5.35), (5.36) and (5.50), we obtain thus

$$\delta\mathsf{M}_n = -\frac{2h^3}{3}[\bar{\lambda}\Delta_\Sigma\omega + 2\bar{\mu}n_{0\beta}n_{0\alpha}\omega_{,\alpha\beta}] + gM_0\widehat{\rho_0\bar{\mu}_3} \tag{5.64}$$

and

$$\delta \mathrm{M}_\tau = -\frac{4\bar{\mu}h^3}{3} n_{0\beta}\tau_{0\alpha}\omega_{,\alpha\beta} \tag{5.65}$$

since $n_{0\beta}\delta_{\beta\alpha 0\alpha} = 1$ and $n_{0\beta}\delta_{\beta\alpha}\tau_{0\alpha} = 0$. On account of eqn (5.50), eqn (5.64) transforms to

$$\delta \mathrm{M}_n = -\frac{2h^3}{3}(\bar{\lambda}\Delta_\Sigma\omega + 2\bar{\mu}\, n_{0\beta}n_{0\alpha}\omega_{,\alpha\beta}) + gM_0\left[\frac{2h^3}{3}\bar{\alpha}M_0\Delta_\Sigma\omega - \widehat{\alpha D\varphi}\right] \tag{5.66}$$

while on account of eqns (5.35), (5.50), (5.36) and (5.48), eqn (5.63) yields

$$\mathscr{D}\frac{\mathrm{d}}{\mathrm{d}n_0}(\Delta_\Sigma\omega) + M_0\frac{\mathrm{d}}{\mathrm{d}n_0}[(1+\bar{\alpha})\widehat{D\varphi}] + \frac{4\bar{\mu}h^3}{3}\frac{\partial}{\partial l_0}(n_{0\beta}\tau_{0\alpha}\omega_{,\alpha\beta}) = 0, \tag{5.67}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}n_0} = \mathbf{n}_{S0}\cdot\nabla_\Sigma. \tag{5.68}$$

If we introduce the two-dimensional curvature tensor of ∂S_0 in Σ_0 by

$$\mathscr{C}_{\beta\alpha} = -\frac{1}{2}n_{0\alpha,\beta}, \tag{5.69}$$

we have

$$\begin{aligned} n_{0\beta}n_{0\alpha}\omega_{,\alpha\beta} &= n_{0\beta}(n_{0\alpha}\omega_{,\alpha})_{,\beta} - n_{0\beta}n_{0\alpha,\beta}\omega_{,\alpha} \\ &= \frac{\mathrm{d}^2\omega}{\mathrm{d}n_0^2} + 2(\mathbf{n}_0\cdot\mathscr{C})\cdot\nabla_\Sigma\omega \end{aligned} \tag{5.70}$$

and

$$\begin{aligned} n_{0\beta}\tau_{0\alpha}\omega_{,\alpha\beta} &= \tau_{0\alpha}(n_{0\beta}\omega_{,\beta})_{,\alpha} - \tau_{0\alpha}n_{0\beta,\alpha}\omega_{,\beta} \\ &= \frac{\partial}{\partial l_0}\left(\frac{\mathrm{d}\omega}{\mathrm{d}n_0}\right) + 2(\boldsymbol{\tau}_0\cdot\mathscr{C})\cdot\nabla_\Sigma\omega. \end{aligned} \tag{5.71}$$

Therefore, eqns (2.51) yield the *statically admissible boundary conditions*

$$\begin{aligned} &\mathscr{D}\frac{\mathrm{d}}{\mathrm{d}n_0}(\Delta_\Sigma\omega) + M_0\frac{\mathrm{d}}{\mathrm{d}n_0}[(1+\bar{\alpha})\widehat{D\varphi}] \\ &+\frac{4\bar{\mu}h^3}{3}\frac{\partial}{\partial l_0}\left[\frac{\partial}{\partial l_0}\left(\frac{\mathrm{d}\omega}{\mathrm{d}n_0}\right) + 2(\boldsymbol{\tau}_0\cdot\mathscr{C})\cdot\nabla_\Sigma\omega\right] = 0 \end{aligned} \tag{5.72}$$

and

$$\frac{2h^3}{3}\left[(\bar{\lambda} - \bar{\alpha}gM_0^2)\Delta_\Sigma\omega + 2\bar{\mu}\left(\frac{\mathrm{d}^2\omega}{\mathrm{d}n_0^2} + 2(\mathbf{n}_0\cdot\mathscr{C})\cdot\nabla_\Sigma\omega\right)\right] + \alpha gM_0\widehat{D\varphi} = 0 \tag{5.73}$$

at regular points along ∂S_0. At angulous points on ∂S_0, eqn (2.52) yields the jump condition

$$\left[\!\left[\frac{\partial}{\partial l_0}\left(\frac{\mathrm{d}\omega}{\mathrm{d}n_0}\right) + 2(\boldsymbol{\tau}_0\cdot\mathscr{C})\cdot\nabla_\Sigma\omega\right]\!\right] = 0. \tag{5.74}$$

The *kinematically admissible boundary conditions* corresponding to eqns (5.72) and (5.73) are easily found by examining the principle of virtual power in its final expression (2.42). Indeed, because of the duality inherent in such a formulation we see that w^* and $\theta^* = -(\partial w^*/\partial n)$ are dual time-rates of the left-hand sides of eqns (5.72) and (5.73) up to a sign. In terms of generalized displacements, and comparing eqns (2.18) and (5.29), we immediately deduce that the kinematically admissible boundary conditions, dual of eqns (5.72) and (5.73), are necessarily

$$\omega = 0 \text{ at } \partial S_0 \tag{5.75}$$

and

$$\frac{d\omega}{dn_0} = 0 \text{ at } \partial S_0 \tag{5.76}$$

since

$$w^* = \frac{\partial \omega^*}{\partial t}, \quad \theta^* = -\frac{\partial w^*}{\partial n_0} = -\frac{\partial}{\partial t}\left(\frac{d\omega^*}{dn_0}\right). \tag{5.77}$$

It remains to determine the magnetic boundary conditions at the upper and lower surfaces of the plate considered as a three-dimensional body. These are obtained by varying eqns (2.57), i.e.

$$\begin{aligned} (\delta \mathbf{n}) \cdot [\mathbf{B}_0] + \mathbf{n}_0 \cdot [\delta \mathbf{B}] &= 0, \\ (\delta \mathbf{n}) \times [\mathbf{H}_0] + \mathbf{n}_0 \times [\delta \mathbf{H}] &= \mathbf{0}, \end{aligned} \tag{5.78}$$

where $\mathbf{n}$ is the unit outward normal either of the upper face, in which case $\mathbf{n}_0 = \mathbf{n}_3$, or of the lower face, then $\mathbf{n}_0 = -\mathbf{n}_3$. In all cases plus and minus upper signs denote the values of a field at the external and the internal face of a surface, respectively. Thus

$$\begin{aligned} &\mathbf{B}_0^+ = \mathbf{H}_0^+, \quad \mathbf{B}_0^- = -({}^L\mathbf{B}_0)^- = -\hat{\chi}_0 \mathbf{M}_0^-, \\ &\mathbf{H}_0^- = +\chi_0^{-1}\mathbf{M}_0^-, \end{aligned} \tag{5.79}$$

after eqns (4.6)–(4.7). Therefore, with

$$\begin{aligned} &[\mathbf{A}] = \mathbf{A}^+ - \mathbf{A}^- \text{ at } x_3 = \pm h, \\ &\mathbf{n} \cdot [\mathbf{A}] = \mathbf{n}_3 \cdot [\mathbf{A}] \text{ at } x_3 = +h, \\ &\mathbf{n} \cdot [\mathbf{A}] = -\mathbf{n}_3 \cdot [\mathbf{A}] \text{ at } x_3 = -h, \\ &\mathbf{M}_0^- = M_0 \mathbf{n}_3 \text{ at } x_3 = +h, \quad \mathbf{M}_0^- = -M_0 \mathbf{n}_3 \text{ at } x_3 = -h \end{aligned} \tag{5.80}$$

we have

$$[\mathbf{H}_0] = \mathbf{H}_0^+ - \mathbf{H}_0^- = \mathbf{H}_0^+ - \chi_0^{-1}\mathbf{M}_0^-. \tag{5.81}$$

Furthermore,

$$\delta \mathbf{B}^+ = \mathbf{h}^+, \quad \delta \mathbf{B}^- = \mathbf{b}^- = \mathbf{h}^- + \rho_0(\bar{\mathbf{u}})^- - \mathbf{M}_0^-(\nabla \cdot \mathbf{u})^- \tag{5.82}$$

in general, with $\delta \mathbf{H} \equiv \mathbf{h}$, after eqn (5.42). In theory, we should account for the variation of the normal $\mathbf{n}_3$ in both eqns (5.78) since there, either $\mathbf{n} = \mathbf{n}_3$ $(x_3 = +h)$ or $\mathbf{n} = -\mathbf{n}_3$ $(x = -h)$. If, however, we make the hypothesis that $\mathbf{H}_0^+ \simeq \mathbf{H}_0^-$ in the initial configuration, which holds true if we discard side effects in the plate (i.e. in fact assimilate the finite plate to a plate of infinite extent insofar as magnetic considerations are concerned), then $[\mathbf{H}_0] \simeq \mathbf{0}$ at $x_3 = \pm h$, and the variation in the normal in eqn $(5.78)_2$ can be discarded. This yields the jump equation

$$\mathbf{n}_{03} \times [\mathbf{h}] = \mathbf{0} \text{ at } x_3 = \pm h. \tag{5.83}$$

The first of eqns (5.78) then yields

$$\begin{aligned} &(\delta \mathbf{n}_3) \cdot \mathbf{M}_0 + \mathbf{n}_{03} \cdot [\mathbf{h}] - \rho_0(\bar{\mu}_n)^- + M_0(\nabla \cdot \mathbf{u})^- = 0 \text{ at } x_3 = +h, \\ &-(\delta \mathbf{n}_3) \cdot \mathbf{M}_0 - \mathbf{n}_{03} \cdot [\mathbf{h}] - \rho_0(\bar{\mu}_n)^- + M_0(\nabla \cdot \mathbf{u})^- = 0 \text{ at } x_3 = -h, \end{aligned} \tag{5.84}$$

for

$$[\mathbf{B}_0] = \mathbf{H}_0^+ - (\chi_0^{-1} - 1)\mathbf{M}_0 = (\mathbf{H}_0^+ - \mathbf{H}_0^-) + \mathbf{M}_0^- \simeq \mathbf{M}_0^-. \tag{5.85}$$

Equations (5.84) also read

$$\begin{aligned} &\left[\frac{\partial \varphi}{\partial x_3}\right] + \rho_0(\bar{\mu}_3)^- - M_0(\nabla \cdot \mathbf{u})^- - M_0(\delta \mathbf{n}_3) \cdot \mathbf{n}_{03} = 0 \text{ at } x_3 = +h \\ &\left[\frac{\partial \varphi}{\partial x_3}\right] + \rho_0(\bar{\mu}_3)^- + M_0(\nabla \cdot \mathbf{u})^- + M_0(\delta \mathbf{n}_3) \cdot \mathbf{n}_{03} = 0 \text{ at } x_3 = -h. \end{aligned} \tag{5.86}$$

The last contributions in these equations vanish as can be seen by variation of $\mathbf{n}_3 \cdot \mathbf{n}_3 = 1$. Using now eqns (5.30) and 5.45), we obtain

$$\begin{aligned} &\left(\frac{\partial \varphi}{\partial x_3}\right)^+ - (1+\alpha)\left(\frac{\partial \varphi}{\partial x_3}\right)^- - M_0[\bar{\alpha}(\nabla_\Sigma \cdot \boldsymbol{\theta}) - \alpha h \Delta_\Sigma \omega] = 0 \text{ at } x_3 = +h \\ &\left(\frac{\partial \varphi}{\partial x_3}\right)^+ - (1+\alpha)\left(\frac{\partial \varphi}{\partial x_3}\right)^- - M_0[\bar{\alpha}(\nabla_\Sigma \cdot \boldsymbol{\theta}) + \alpha h \Delta_\Sigma \omega] = 0 \text{ at } x_3 = -h. \end{aligned} \tag{5.87}$$

In addition we must have

$$[\varphi] = 0 \text{ at } x_3 = \pm h, \text{ and } \varphi \to 0 \text{ at infinity from the plate.} \tag{5.88}$$

6. SUMMARY OF BENDING EQUATIONS AND COMMENTS

On neglecting the coupling with $\boldsymbol{\theta}$—which is usually done in the Kirchhoff–Love theory of thin plates by setting $\boldsymbol{\theta} = \mathbf{0}$ at the start—we finally have the following coupled mechanical-magnetic equilibrium equations for static perturbations of isotropic soft-ferromagnetic elastic thin plane plates:

- *in the medium plane* S_0 *of the plate*:

$$\mathscr{D}\,\Delta_\Sigma \Delta_\Sigma \omega - 2hM_0^2(1+\bar{\alpha})\Delta_\Sigma \omega + M_0[(1+\bar{\alpha})(\Delta_\Sigma \widehat{D\varphi} + (1+\alpha)D^2\varphi = 0 \tag{6.1}$$

- *at regular points of its contour* ∂S_0, either

$$\begin{aligned} &\mathscr{D}\frac{\mathrm{d}}{\mathrm{d}n_0}(\Delta_\Sigma \omega) + M_0 \frac{\mathrm{d}}{\mathrm{d}n_0}[(1+\bar{\alpha})\,\widehat{D\varphi}] \\ &+ \frac{4\bar{\mu}h^3}{3}\frac{\partial}{\partial l_0}\left[\frac{\partial}{\partial l_0}\left(\frac{\mathrm{d}\omega}{\mathrm{d}n_0}\right) + 2(\boldsymbol{\tau}_0 \cdot \mathbf{K}) \cdot \nabla_\Sigma \omega\right] = 0 \end{aligned} \tag{6.2}$$

or

$$\omega = 0, \tag{6.3}$$

and either

$$\frac{2h^3}{3}\left\{(\bar{\lambda} - \bar{\alpha} g M_0^2)\Delta_\Sigma \omega + 2\bar{\mu}\left[\frac{d^2\omega}{dn_0^2} + 2(\mathbf{n}_0 \cdot \mathscr{C}) \cdot \nabla_\Sigma \omega\right]\right\} + \alpha g \widehat{M_0 D\varphi} = 0 \tag{6.4}$$

or

$$\frac{d\omega}{dn_0} = 0, \tag{6.5}$$

- *at angulous points along* ∂S_0:

$$\left[\frac{\partial}{\partial l_0}\left(\frac{d\omega}{dn_0}\right) + 2(\tau_0 \cdot \mathscr{C}) \cdot \nabla_\Sigma \omega\right] = 0, \tag{6.6}$$

- *in the three-dimensional volume* $\mathscr{D}_0$ *of the plate*:

$$(1 + \chi_0)\Delta_\Sigma \varphi + (1 + \alpha)D^2\varphi = M_0(1 + \bar{\alpha})\Delta_\Sigma \omega, \tag{6.7}$$

- *on the upper face* $\{x_3 = +h\}$ *of the plate*

$$\left(\frac{\partial\varphi}{\partial x_3}\right)^+ - (1 + \alpha)\left(\frac{\partial\varphi}{\partial x_3}\right)^- + M_0\alpha h\, \Delta_\Sigma \omega = 0, \quad \varphi^+ = \varphi^-, \tag{6.8}$$

- *on the lower face* $\{x_3 = -h\}$ *of the plate*:

$$\left(\frac{\partial\varphi}{\partial x_3}\right)^+ - (1 + \alpha)\left(\frac{\partial\varphi}{\partial x_3}\right)^- - M_0\alpha h\, \Delta_\Sigma \omega = 0, \quad \varphi^+ = \varphi^-; \tag{6.9}$$

- *outside the plate* $\{|x_3| > h\}$:

$$\nabla^2\varphi = 0, \quad \varphi \to 0 \text{ as } |x_3| \to \infty. \tag{6.10}$$

We recall that ∇_Σ and Δ_Σ are the two-dimensional gradient and Laplacian operators in the medium plane of the plate, $\mathbf{n}_0$ is the external normal to the plate in this plane, d/dn_0 is the corresponding derivative along this normal, τ_0 is the unit tangent to the contour of the plate and $\partial/\partial l_0$ is the corresponding derivative, $\mathscr{C}$ is the curvature tensor of the contour, $D = \partial/\partial x_3$ where x_3 is the normal co-ordinate to the plate, and ∇^2 is the usual three-dimensional Laplacian operator. The symbolism $\widehat{(..)}$ and $\widehat{\widehat{(..)}}$ represent averages and averaged moments throughout the thickness of the plate, i.e.

$$\hat{A} = \int_{-h}^{+h} A(., x_3)\, dx_3, \quad \hat{\hat{A}} = \int_{-h}^{+h} A(., x_3)x_3\, dx_3. \tag{6.11}$$

Finally, ω is the deflection of the plate and φ is the perturbation magnetostatic potential. We have set

$$\alpha = [(\hat{\chi}_0 + \beta) - 1]^{-1}, \quad \chi_0 = (\hat{\chi}_0 - 1)^{-1}, \quad \bar{\alpha} = \alpha(1 + g), \tag{6.12}$$

where g is a coefficient of induced piezomagnetism, β is a magnetic anisotropy constant, $\hat{\chi}_0$ is the isotropic magnetic constant, and $\bar{\lambda}$ and $\bar{\mu}$ are stiffened elastic moduli. $\mathscr{D}$ is the flexural rigidity of the plate of thickness $2h$. A remark is in order concerning the value of this flexural rigidity. In the classical bending theory of thin flat plates (see, e.g. Saada[18], p. 525), this quantity is given by $D = 2h^3/3(1 - \nu^2)$ in terms of Poisson's ratio ν (on account of the fact that the plate has thickness $2h$ and not h). If we make abstraction of the alteration brought in by the initial magnetization, in the present case we have (see eqn 5.31)

$$\mathscr{D} \rightarrow D = \frac{2h^3}{3}(\lambda + 2\mu) = \frac{2h^3}{3}\frac{E}{(1-\nu^2)}\delta \qquad (6.13)$$

with

$$\delta = (1-\nu)^2/(1-2\nu), \qquad (6.14)$$

as is readily checked. For instance, for a material having a Poisson ratio 1/4, $\delta = 9.8 \simeq 1.12$, while $\delta = 1$ in the usual theory. This discrepancy arises from the differing basic assumptions as regards the state of stresses or strains in the plate. While plane stresses are assumed in the classical theory, we have assumed plane strains (see eqns 2.22). In doing so we slightly overestimate the flexural rigidity as compared to the classical case. However, the magnetic correction in eqn (5.53) has for effect to lower the value, and this probably by a few per cent for sufficiently intense bias magnetic fields, so that in the end the flexural rigidity will not be much different from that of the classical case; for estimates of magnetic corrections to mechanical properties, see Maugin[16], Sec. 2.2.

Final remarks concern both the theoretical physical bases of the present approach and practical situations. The physical bases on which are built the fundamental assumptions in the beginning of Section 2 are those explained at length in a long review paper[12]. The introduction of particular expressions for the volume and surface ponderomotive forces, $\mathbf{f}^{em}$ and $\mathbf{T}^{em}$ in eqn (2.3), is, as well-known, a matter of personal choice. Certain authors rely on peculiar formulations of electrodynamics (see [5, 7, 9]), others prefer to let these quantities somewhat undefined so that an automatic adjustment of matter and electromagnetic-field contributions occurs through nonlinear constitutive equations (see, e.g. [19])—this ultimately follows from the fact that electromagnetic fields in themselves do not constitute a closed thermodynamical system (see the discussion in Ref. [20]). Our choice, which differs from others'[5, 7, 9] follows from expressions accepted in mathematical physics (an apparently best unified treatment of coupled electromagneto-mechanical effects is given in a series of papers, Ref. [19]). Other enlightening discussions on this problem are to be found in Hutter and Van de Ven[21] as also in Van de Ven[8]. Various choices manifest in various electromagnetic "source" contributions in the mechanical equations (6.1) through (6.4). Only a complete treatment of the eigenvalue problem connected with eqns (6.1), (6.7) and (6.10) and the associated boundary conditions for plates of various shapes can shed light on the different outcomes. This treatment will be given in a forthcoming companion paper. The complexity of the solution obviously depends on the shape of the plate (circular or rectangular) and the nature of the mechanical boundary conditions. The value of the critical "buckling" magnetic field will follow from the analysis of the bending modes of the plate. Then the difference in various primitive choices of electromagnetic formulation will materialize in various values of this buckling value. An important remark here concerns the methodology (use of the principle of virtual power) and the consequently simultaneous obtaining of boundary conditions. Other types of boundary conditions (elastically restrained or elastically clamped plates) will also be introduced to represent physical reality in a more satisfactory manner.

We must also notice by way of conclusion that the initial magnetic field has been here considered as spatially uniform. The variational technique introduced in Section 5 with a view to deducing the equations governing small fields superimposed on biases can be generalized without difficulty, but at the price of lengthy calculations, to the case of nonuniform initial magnetic fields. The variation technique then is the so-called Lagrangian variation (variation of the nonlinear equations at fixed Lagrangian or material co-ordinates)—see ([16], p. 339) for a general definition of this variation in nonuniform electromagneto-mechanical initial configurations. The introduction of this additional complexity would be useful in studying the influence of a weak gradient in the initial magnetic field, a situation which certainly prevails in many experimental settings and practical devices where the realization of a true spatially uniform magnetic field is not feasible. If such is the case, this nonuniformity will give rise to additional electromagnetic source terms in eqns (6.1)–(6.4), and this may explain certain discrepancies between experimental and theoretical values of magnetoelastic buckling fields. This could be achieved at the price of lengthier expressions for eqns (6.1)–(6.6) and great complications in the buckling study which is already quite involved in the case of plates of rectangular shapes. The above-obtained equations therefore hold good only for very weakly nonuniform initial magnetic fields, for instance, initial magnetic fields which vary over space with a typical length scale which is much larger than the wavelength of observed bending modes. Since this wavelength can be assumed to be at most of the order of twice the widest dimension L of the piece of structure, with $|\nabla \mathbf{H}_0| \simeq H_0/L_H$, we must have $L_H \gg 2H$.

REFERENCES

1. F. C. Moon, Problems in magneto-solid mechanics. In: *Mechanics Today*, (Edited by S. Nemat-Nasser), Vol. 4, pp. 307–390. Pergamon Press, New York (1978).
2. F. C. Moon and Y-H. Pao, Magnetoelastic buckling of a thin plate. *J. Appl. Mech*, **35**, 53–58 (1968).
3. F. C. Moon and Y-H. Pao, Vibration and dynamic instability of a beam-plate in a transverse magnetic field. *J. Appl. Mech.* **36**, 92–100 (1969).
4. D. V. Wallerstein and M. O. Peach, Magnetoelastic buckling of beams and thin plates of magnetically soft material. *J. Appl. Mech.* **39**, 451–455 (1972).
5. Y-H. Pao and C-S. Yeh, A linear theory for soft ferromagnetic elastic solids. *Int. J. Engng Sci.* **11**, 415 (1973).
6. J. M. Dalrymple, M. O. Peach and G. L. Vligelahn, Magnetoelastic buckling of thin magnetically soft plates in cylindrical mode. *J. Appl. Mech.* **41**, 145–150 (1974).
7. A. A. F. van de Ven, Interaction of electromagnetic and elastic fields. Ph.D. Thesis, Eindhoven Technological University, Eindhoven, Chapter IX, The Netherlands (1975).
8. A. A. F. van de Ven, Magnetoelastic buckling of thin plates in a uniform transverse magnetic field. *J. Elasticity* **8**, 297–312 (1978).
9. H. Parkus, Application of electromagnetic interaction theory. In: *Electromagnetic Interactions in Elastic Solids* (Edited by H. Parkus), pp. 363–415. Springer-Verlag, Wien (1979).
10. G. A. Maugin, A continuum approach to magnon–phonon couplings—I. *Int. J. Engng Sci.* **17**, 1073–1091 (1979).
11. G. A. Maugin, Sur la dynamique des milieux déformables magnétisés avec Spin Magnétique. *J. Mécanique* **13**, 75–96 (1974).
12. G. A. Maugin, The principle of virtual power in continuum mechanics: application to coupled fields. *Acta Mechanica* **35**, 1–70 (1980).
13. P. Germain, Course of Mechanics at Ecole Polytechnique, Paris, Year 1975–76, Chapter IV, Editions de l'Ecole Polytechnique, Paris (1976).
14. G. A. Maugin, A continuum approach to magnon–phonon couplings—II. *Int. J. Engng Sci.* **17**, 1093–1108 (1979).
15. G. A. Maugin and J. Pouget, Electroacoustic equations for one-domain ferroelectric bodies. *J. Acoust. Soc. Amer.* **68**, 575–587 (1980).
16. G. A. Maugin, Wave motion in magnetizable deformable solids. *Int. J. Engng Sci.* **19**, 321–388 (1981).
17. G. A. Maugin, Sur l'objectivité des entités électromagnétiques dans un corps en mouvement à l'approximation galiléenne. *C.R. Acad. Sci. Paris* **289B**, 205–208 (1979).
18. A. S. Saada, *Elasticity: Theory and Applications*. Pergamon, New York (1974).
19. G. A. Maugin, On the covariant formulation of the relativistic electrodynamics of continua—I–IV. *J. Math. Phys.* **19**, 1198–1205, 1206–1211, 1212–1219, 1220–1226 (1978).
20. G. A. Maugin, Further comments of the equivalence of Abraham's, Minkowski's, and others' electrodynamics. *Can. J. Phys.* **58**, 1163–1170 (1980).
21. K. Hutter and A. A. F. van de Ven, Field matter interactions in thermoelastic solids. *Lecture Notes in Physics* Vol. 88. Springer-Verlag, Berlin (1978).